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The Unstable Behaviour of Reciprocally Rotating Shafts in Textile Machines

Abstract
The stability of the reciprocal rotary motion of a motor-driven elastic shaft is studied in this paper. It is found that unstable critical behaviour arises when the average value of the second power of angular velocity is close to the second power of the circular frequency of natural vibrations.

Key words: *reciprocal, rotating shaft, textile machines, stability at rotary motion.*

Introduction

Reciprocally rotating shafts are quite common in textile machines such as looms [1], carding machines [2], and sewing machines [3]. The torsional vibrations of such systems have been investigated in work [4]. The resonance of flexural vibration has been studied in works [5]. In the case of rotating shafts, unstable behaviour arises when the angular velocity is equal to the circular frequency of natural vibrations [6]. This equality is the reason that the so-called critical unstable behaviour is often confused with the resonance behaviour of flexural vibration. In fact both phenomena are essentially different. This is quite clear in the case of the reciprocally rotating shaft, where critical speed is different from resonance frequencies.

Equations of motion

The system considered, shown in Figure 1, consists of the following elements: (I) an elastic shaft with unit length mass μ, (II) an attached inertia B, (III) a motor of inertia A and torque M and (IV) the mechanism transforming the rotary motion of element A to the rotary reciprocal motion of element B. This mechanism is not shown, and it may be either a cam or a linkage mechanism. Torsional vibrations are not included in this study [4]. The equation of motion governing the flexural vibrations [5] of an oscillating shaft has the form (1). Here, α is the angular coordinate of a shaft, EI denotes the shaft stiffness, χ the damping coefficient, v and w are components of the transverse displacement of the shaft element, and v_0 and w_0 are components of the shaft eccentricity, while ξ is a coordinate along the axis shaft.

Let us denote an eigenfunction representing the shaft transverse deflection of the first mode of natural vibration by Ξ(ξ). Substituting expressions (2) into Equations (1) (see page 101), multiplying by the function Ξ and then integrating the resultant expression, we obtain a set of ordinary differential Equations (3), where

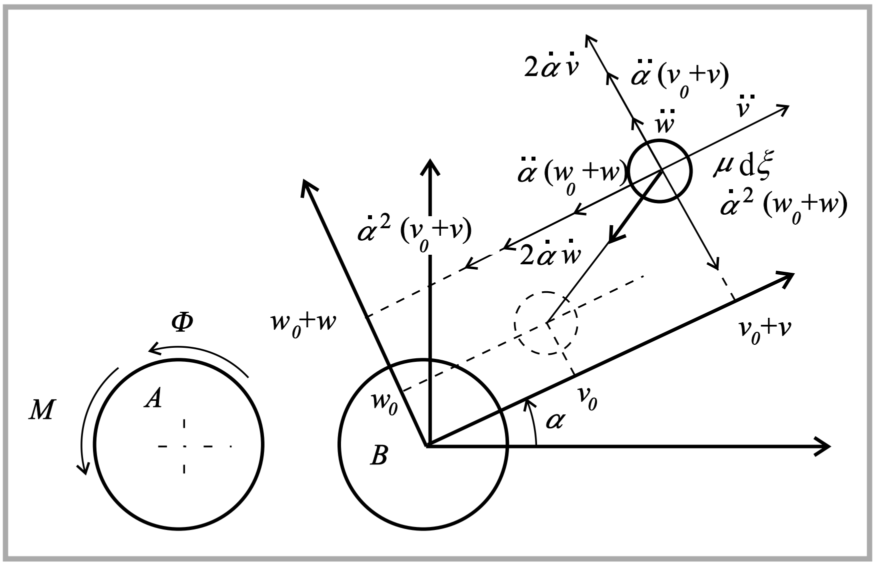

Figure 1. *Accelerations of reciprocally rotating shaft driven by the motor torque M through mechanism (not shown) connecting elements A and B.*

$$\mu\left[\frac{\partial^2 v}{\partial t^2}-\frac{\mathrm{d}^2\alpha}{\mathrm{d}t^2}(w+w_0)-\left(\frac{\mathrm{d}\alpha}{\mathrm{d}t}\right)^2(v+v_0)-2\frac{\mathrm{d}\alpha}{\mathrm{d}t}\frac{\partial w}{\partial t}\right]+EI_1\left(1+\chi\frac{\partial}{\partial t}\right)\frac{\partial^4 v}{\partial\xi^4}=0$$
$$\mu\left[\frac{\partial^2 w}{\partial t^2}+\frac{\mathrm{d}^2\alpha}{\mathrm{d}t^2}(v+v_0)-\left(\frac{\mathrm{d}\alpha}{\mathrm{d}t}\right)^2(w+w_0)+2\frac{\mathrm{d}\alpha}{\mathrm{d}t}\frac{\partial v}{\partial t}\right]+EI_2\left(1+\chi\frac{\partial}{\partial t}\right)\frac{\partial^4 w}{\partial\xi^4}=0 \tag{1}$$

$$a_y=\frac{\mathrm{d}^2y}{\mathrm{d}t^2}+2\frac{\mathrm{d}\alpha}{\mathrm{d}t}\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\mathrm{d}^2\alpha}{\mathrm{d}t^2}(x_0+x)-\left(\frac{\mathrm{d}\alpha}{\mathrm{d}t}\right)^2(y_0+y)$$
$$a_x=\frac{\mathrm{d}^2x}{\mathrm{d}t^2}-2\frac{\mathrm{d}\alpha}{\mathrm{d}t}\frac{\mathrm{d}y}{\mathrm{d}t}-\frac{\mathrm{d}^2\alpha}{\mathrm{d}t^2}(y_0+y)-\left(\frac{\mathrm{d}\alpha}{\mathrm{d}t}\right)^2(x_0+x). \tag{4}$$

$$m=\int_0^l\mu\Xi^2\mathrm{d}\xi,\quad k_1=\int_0^l\Xi EI_1\frac{\partial^4\Xi}{\partial\xi^4}\mathrm{d}\xi,\quad c_1=\int_0^l\Xi EI_1\chi\frac{\partial^4\Xi}{\partial\xi^4}\mathrm{d}\xi$$
$$k_2=\int_0^l\Xi EI_2\frac{\partial^4\Xi}{\partial\xi^4}\mathrm{d}\xi,\quad c_2=\int_0^l\Xi EI_2\chi\frac{\partial^4\Xi}{\partial\xi^4}\mathrm{d}\xi,\quad x_0=\int_0^l\Xi v_0\mathrm{d}\xi,\quad x_0=\int_0^l\Xi w_0\mathrm{d}\xi \tag{5}$$

Equations 1, 4, and 5.

m is reduced mass, k reduced stiffness, c the reduced damping coefficient, x and y are components of the reduced transverse displacement of the shaft element, x_0 and y_0 are components of the reduced shaft eccentricity, ω_1 and ω_2 are circular frequencies of natural vibrations.

$$\Xi = \Xi(\xi)\,,\; v = x\Xi,\; w = y\,\Xi \tag{2}$$

$$\begin{aligned} ma_x + c_1\frac{\mathrm{d}x}{\mathrm{d}t} + k_1 x = 0 \\ ma_y + c_2\frac{\mathrm{d}y}{\mathrm{d}t} + k_2 y = 0 \end{aligned} \tag{3}$$

$$\omega_1 = \sqrt{\frac{k_1}{m}},\quad \omega_2 = \sqrt{\frac{k_2}{m}} \tag{6}$$

Using the principle of virtual works (7) and relationships (3, 8, 9, 10, 11), we obtain equation (12) governing the rotary motion of the shaft. Here, Φ is an angular coordinate of the driving shaft, M_A the torque acting on the driving shaft and M_B the torque acting on the driven shaft.

$$M_A \mathrm{d}\Phi + M_B \mathrm{d}\alpha = 0 \tag{7}$$

$$M_A = -A\frac{\mathrm{d}^2\Phi}{\mathrm{d}t^2} + M \tag{8}$$

The driving torque M of the motor can be found from differential equation (13). In this equation T denotes the time constant of the motor, C_m the stiffness of the motor characteristic and Ω is the idle speed of the motor.

$$T\frac{\mathrm{d}M}{\mathrm{d}t} = C_m\left(\Omega - \frac{\mathrm{d}\Phi}{\mathrm{d}t}\right) - M \tag{13}$$

Resonance behaviour

The resonance behaviour of a reciprocally rotating shaft has been explained in work [5]. Using complex numbers for $k_1 = k_2$, Equations (3, 4) can be rewritten in a more convenient form (14.a) and for $c = 0$ a further simplified form may be found (14.b).

If the angle of shaft oscillation is given explicitly in time by the sine function, then equation (14.b) can be put into form (15).

From equation (15), it follows that: (I) resonance resulting from forces responsible for angular acceleration takes place when the circular frequency of periodic motion is equal to that of natural vibration $\Omega = \omega_0$, (II) resonance resulting from centrifugal forces takes place when the circular frequency of periodic motion is equal to half that of natural vibration $\Omega = \omega_0/2$.

Critical behaviour

If the average value of the mass displacement is large compared to the amplitude of vibrations, then the set of equations (3,4) may be simplified by replacing its terms with their average values (16). A large average displacement x may be expected when the denominator of quotient (17) approaches zero. This takes place when the average value of the second power of angular velocity $\mathrm{d}\alpha/\mathrm{d}t$ is close to the second power of the circular frequency of natural vibrations ω_0 (18). Multiplying Equation (18) by mass displacement, this condition may be expressed in terms of centrifugal and restoring forces (19). Similar results can be obtained by considering the displacement in direction y. Finally, from equation (18), the formula for calculating approximate value of critical of angular velocity $\mathrm{d}\Phi/\mathrm{d}t$ is found to have the form (20).

$$M_B = -B\frac{\mathrm{d}^2\alpha}{\mathrm{d}t^2} + ma_x(y_0 + y) - ma_y(x_0 + x) \tag{9}$$

$$M_B = -B\frac{\mathrm{d}^2\alpha}{\mathrm{d}t^2} - \left(c_1\frac{\mathrm{d}x}{\mathrm{d}t} + k_1 x\right)(y_0 + y) + \left(c_2\frac{\mathrm{d}y}{\mathrm{d}t} + k_2 y\right)(x_0 + x) \tag{10}$$

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = \frac{\mathrm{d}\alpha}{\mathrm{d}\Phi}\frac{\mathrm{d}\Phi}{\mathrm{d}t},\qquad \frac{\mathrm{d}^2\alpha}{\mathrm{d}t^2} = \frac{\mathrm{d}^2\alpha}{\mathrm{d}\Phi^2}\left(\frac{\mathrm{d}\Phi}{\mathrm{d}t}\right)^2 + \frac{\mathrm{d}\alpha}{\mathrm{d}\Phi}\frac{\mathrm{d}^2\Phi}{\mathrm{d}t^2} \tag{11}$$

$$\left[A + B\left(\frac{\mathrm{d}\alpha}{\mathrm{d}\Phi}\right)^2\right]\frac{\mathrm{d}^2\Phi}{\mathrm{d}t^2} + \left[B\frac{\mathrm{d}^2\alpha}{\mathrm{d}\Phi^2}\left(\frac{\mathrm{d}\Phi}{\mathrm{d}t}\right)^2 + \left(c_1\frac{\mathrm{d}x}{\mathrm{d}t} + k_1 x\right)(y_0 + y) - \left(c_2\frac{\mathrm{d}y}{\mathrm{d}t} + k_2 y\right)(x_0 + x)\right]\frac{\mathrm{d}\alpha}{\mathrm{d}\Phi} = M \tag{12}$$

$$\begin{gathered} z = x + iy,\quad i = \sqrt{-1},\quad c_1 = c_2 = c,\quad k_1 = k_2 = k \\ m\left[\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} + 2i\frac{\mathrm{d}\alpha}{\mathrm{d}t}\frac{\mathrm{d}z}{\mathrm{d}t} + i\frac{\mathrm{d}^2\alpha}{\mathrm{d}t^2}(z_0 + z) - \left(\frac{\mathrm{d}\alpha}{\mathrm{d}t}\right)^2(z_0 + z)\right] + c\frac{\mathrm{d}z}{\mathrm{d}t} + kz = 0 \end{gathered} \tag{14a}$$

$$\begin{gathered} z = e^{-i\alpha}Z,\quad \omega_0 = \sqrt{\frac{k}{m}},\quad c = 0 \\ \frac{\mathrm{d}^2 Z}{\mathrm{d}t^2} + \omega_0^2 Z = e^{i\alpha}\left[-i\frac{\mathrm{d}^2\alpha}{\mathrm{d}t^2}z_0 + \left(\frac{\mathrm{d}\alpha}{\mathrm{d}t}\right)^2 z_0\right] \end{gathered} \tag{14b}$$

$$\begin{gathered} \alpha = \frac{\alpha_0}{2}\sin\Omega t,\quad e^{i\alpha} \cong 1 + i\alpha = 1 + i\frac{\alpha_0}{2}\sin\Omega t \\ \frac{\mathrm{d}^2 Z}{\mathrm{d}t^2} + \omega_0^2 Z = z_0\left[i\frac{\alpha_0}{2}\Omega^2\sin\Omega t + \left(\frac{\alpha_0}{2}\Omega\cos\Omega t\right)^2\right]\left(1 + i\frac{\alpha_0}{2}\sin\Omega t\right) \end{gathered} \tag{15}$$

$$k_1 x - m\left(\frac{\mathrm{d}\alpha}{\mathrm{d}t}\right)^2(x_0 + x) = 0 \tag{16}$$

$$x = \frac{\left(\frac{\mathrm{d}\alpha}{\mathrm{d}t}\right)^2 x_0}{\omega_0^2 - \left(\frac{\mathrm{d}\alpha}{\mathrm{d}t}\right)^2},\quad \omega_1 = \sqrt{\frac{k_1}{m}} \tag{17}$$

$$\operatorname{average}\left(\left(\frac{\mathrm{d}\alpha}{\mathrm{d}t}\right)^2\right) \cong \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\mathrm{d}\alpha}{\mathrm{d}\Phi}\right)^2 \mathrm{d}\Phi\left(\frac{\mathrm{d}\Phi}{\mathrm{d}t}\right)^2 \cong \omega_1^2 \tag{18}$$

$$\operatorname{average}\left(\left(\frac{\mathrm{d}\alpha}{\mathrm{d}t}\right)^2\right)xm \cong xk_1 \tag{19}$$

$$\left(\frac{d\Phi}{dt}\right)_{cr} = \Omega_{cr} \cong \frac{\omega_{\min}}{\sqrt{\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{d\alpha}{d\Phi}\right)^2 d\Phi}},\quad \omega_{\min} = \min(\omega_1, \omega_2) \tag{20}$$

Equations 9, 10, 11, 12, 14a, 14b, 15, 16, 17, 18, 19, and 20.

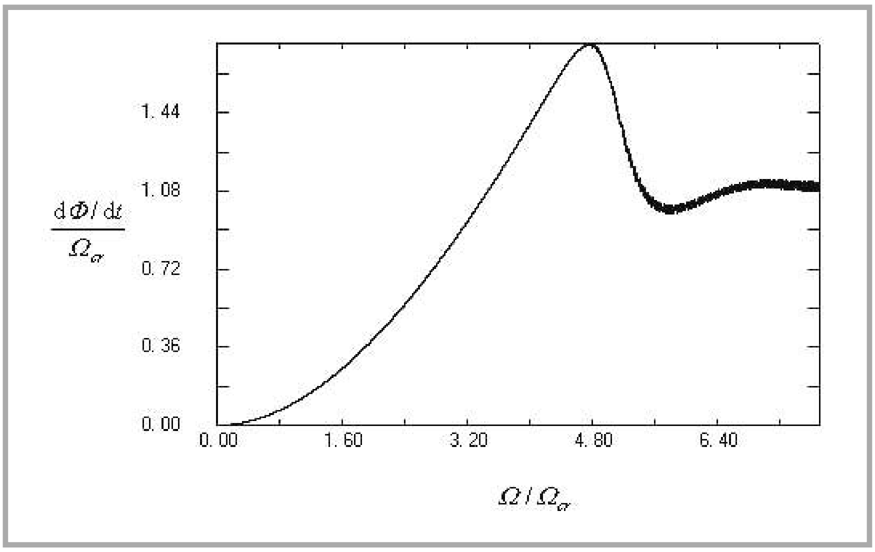

Figure 2. *The average angular velocity dΦ/dt of a driving shaft as a function of increasing the idle speed of the motor Ω for* $\omega_1=\omega_2=100\pi/3$, $c_1/(mk_1)^{0.5}=c_2/(mk_2)^{0.5}=10$.

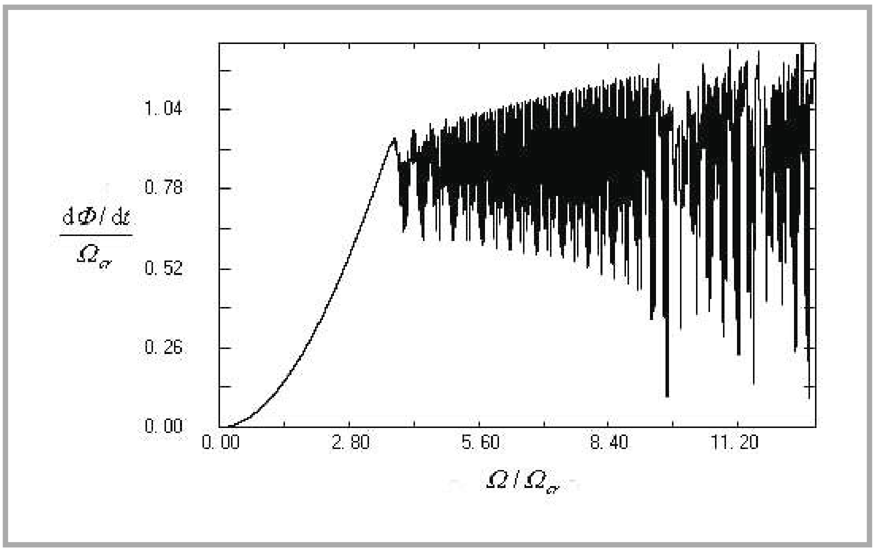

Figure 3. *The average angular velocity dΦ/dt of a driving shaft as a function of increasing the idle speed of the motor Ω for* $\omega_1=100\pi/3$, $\omega_2=500\pi/3$, $c_1/(mk_1)^{0.5}=0.1$, $c_2/(mk_2)^{0.5}=0.1$.

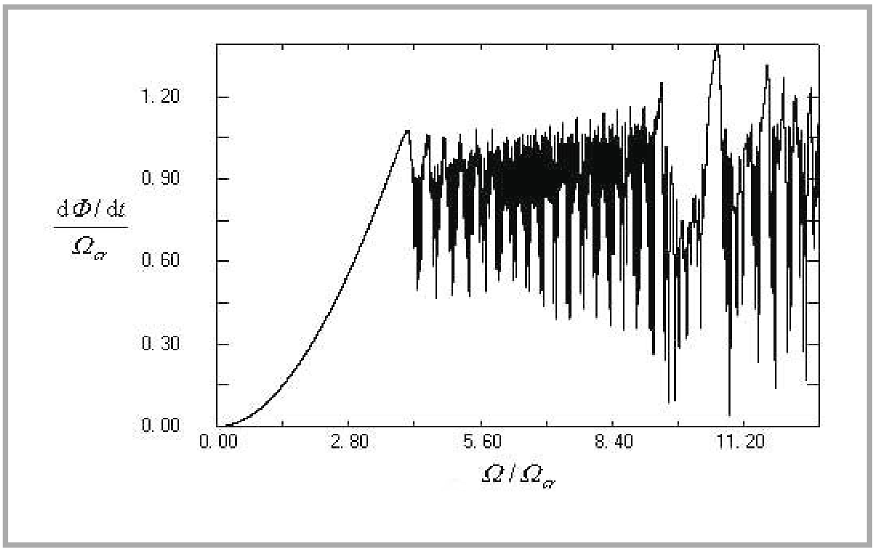

Figure 4. *The average angular velocity dΦ/dt of a driving shaft as a function of increasing the idle speed of the motor Ω for* $\omega_1=500\pi/3$, $\omega_2=100\pi/3$, $c_1/(mk_1)^{0.5}=0.1$, $c_2/(mk_2)^{0.5}=0.1$.

Numerical results and discussion

Taking the dependence between the motion of driving and driven shaft of the form (21)

$$\alpha(\Phi) = \alpha_0/2 \sin(\Phi) \qquad (21)$$

the set of non-linear Equations (3, 4, 12, 13) was solved numerically using the Runge-Kutta method, taking the time step $\Delta t = 2\pi/\max(\omega_1,\omega_2,\Omega)/18000$ and changing the idle speed Ω of the motor with a step $\Delta\Omega = \Omega_{cr}/2500$ at every $100\Delta t$. The initial conditions were taken as: $t = 0$, $x = y = 0$, $dx/dt = dy/dt = 0$, $\Phi = 0$, $d\Phi/dt = 0$, $M = 0$. The results for $\alpha_0 = \pi/4$, $A = 10$, $B = 0.01$, $m = 10$, $x_0 = 0.01$, $y_0 = 0$, $C_m = 10$, $T = 1/\Omega_{cr}$, $\Omega_{cr} = \min(\omega_1,\omega_2)/(\alpha_0/8^{0.5})$ are shown in Figures 2 - 4.

From Figures 2 - 4 it may be seen that the increase in Ω causes the increase of the angular velocity $d\Phi/dt$ until it passes the critical value Ω_{cr}. Further increase of Ω is not followed by the increase of the velocity, but it results in the unstable behaviour of the system. It should be noted that this behaviour requires internal damping to be present in a system [6, 7].

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