Bohuslav Neckář,
Dipayan Das

Department of Textile Structures, Textile Faculty, Technical University of Liberec,
Hálkova 6, Liberec 1, 461 17, Czech Republic
E-mail: bohuslav.neckar@vslib.cz

Mechanics of Parallel Fibre Bundles

Abstract
This theoretical work deals with the mechanics of parallel fibre bundles, on the basis of the fact that each fibre in the bundle possesses different tensile behaviours. As a consequence of this, the tensile behaviour of a blended fibre bundle is found to be different than that obtained from Hamburger's theory [3]. It is also observed that the average force per fibre in the bundle, the breaking force utilisation coefficient, and the breaking strain utilisation coefficient depend only on the coefficient of variation of the fibre breaking strain.

Key words: *parallel fibre bundle, random fibre breaking points, similar force-strain relation, symmetrical breaking force, utilisation coefficient, coefficient of variation of fibre breaking strain.*

Introduction

The tensile behaviour of parallel fibre bundles has always been an interesting topic for textile researchers. It is well known that the tensile properties of a fibre bundle are greatly influenced by the tensile properties of the constituent fibres which form the bundle. Therefore, a complete understanding of the mechanism of translation of stress-strain curves of the constituent fibres into the tensile properties of the bundle is of great importance. In this regard, perhaps the simplest theoretical model assumes that all of the constituent fibres of a fibre bundle follow the same stress-strain curve and have the same breaking stress and breaking strain. Modelling the tensile behaviour of such a bundle is a trivial task. The tensile properties of a multi-component fibre bundle, where all the components relate to the same fibre material, were first formulated by Sinitsin [1]; subsequently, those formulas were found to be in good agreement with the actual results of spun yarns produced from the mixing of different varieties of Egyptian cotton fibres of different lengths and fineness [2]. A more complicated case concerns a blended fibre bundle consisting of multiple components, where one component has a different stress-strain behaviour than that of the other, but all the constituent fibres within a particular component have the same breaking stress and breaking strain. This case was first studied by Hamburger [3] on a two-component blended yarn. Later on, this study was extended to a three-component blended yarn by Żurek [4]. However, the experimental investigation carried out by Kemp & Owen [5] showed that Hamburger's theory was at variance with the facts. Of course, a real fibre bundle consists of fibres possessing different stress-strain behaviours, a fact which was not considered in Hamburger's theory. Taking this fact into consideration, a new theory on the tensile behaviour of parallel fibre bundles has been developed, and is presented in this paper with many examples.

Theory and examples

We deem a fibre bundle to consist of a large number of straight and mutually parallel fibres, and each of these fibres is gripped by both jaws of a tensile tester during the tensile testing of the bundle. The tensile behaviour of this bundle will be discussed in the following sections under some assumptions.

Assumption of random character of fibre breaking points

The breaking points (P,a) of the fibres, shown schematically by the symbol '•' in Figure 1, are random; their distribution is characterised by the joint probability density function $u(P,a)$, where $P \in \langle P_{min}, P_{max} \rangle$ is the fibre breaking force and a $\langle a_{min}, a_{max} \rangle$ is the fibre breaking strain. The average breaking point of fibres, shown by the symbol 'o' in Figure 1, is characterised by average fibre breaking force $\overline{P}$ and average fibre breaking strain $\overline{a}$ as follows:

$$\overline{P} = \int_{a_{min}}^{a_{max}} \int_{P_{min}}^{P_{max}} P u(P,a) dP da \qquad (1)$$

$$\overline{a} = \int_{a_{min}}^{a_{max}} \int_{P_{min}}^{P_{max}} a u(P,a) dP da \qquad (2)$$

The marginal probability density function of the fibre breaking strain $g(a)$ is given by

$$g(a) = \int_{P_{min}}^{P_{max}} u(P,a) dP \qquad (3)$$

and the corresponding distribution function $G(a)$ is

$$G(a) = \int_{a_{min}}^{a} g(\alpha) d\alpha \qquad (4)$$

Substituting Equation (3) for (2), the average fibre breaking strain takes another form, as follows:

$$\overline{a} = \int_{a_{min}}^{a_{max}} a g(a) da \qquad (5)$$

From the theory of probability, we know that

$$u(P,a) dP da = \psi(P|a) dP g(a) da,$$
or
$$\psi(P|a) = u(P,a)/g(a) \qquad (6)$$

where $\psi(P|a)$ is the conditional probability density function of the fibre breaking force P at a given fibre breaking strain a. Using Equation (6), the conditional average fibre breaking force at a given fibre breaking strain $\overline{P(a)}$ is obtained as follows:

$$\overline{P(a)} = \int_{P_{min}}^{P_{max}} P \psi(P \mid a) dP = \frac{1}{g(a)} \int_{P_{min}}^{P_{max}} P u(P,a) dP \qquad (7)$$

This is shown by symbol 'Δ' in Figure 1.

Assumption of similarity in force S - strain ε relation of fibres

The fibres have a similar force-strain relation $S = S(\varepsilon)$, such that at or before a fibre breaks ($\varepsilon \leq a$), its tensile behaviour follows the relation $S(\varepsilon) = k\overline{S}(\varepsilon)$, where $\overline{S}(\varepsilon)$, as we call it, is an average function characterising the average force-strain relation of fibres, and k is a fibre param-

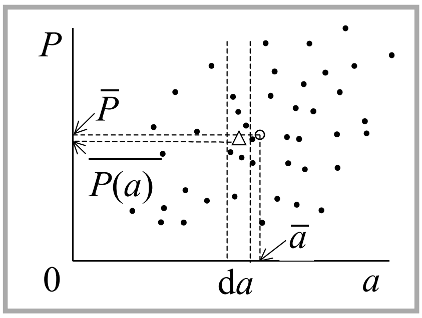

Figure 1. *Distribution of fibre breaking points.*

eter. Here we introduce the convention that the average function passes through the average breaking point of fibres, as shown in Figure 2. Hence the following expression is obvious:

$$\bar{P} = \bar{S}(\bar{a}) \tag{8}$$

Thus the following relation holds at the breaking point of each fibre:

$$P = S(a) = k\bar{S}(a), \text{ or } k = P/\bar{S}(a) \tag{9}$$

So the force-strain relation of a general fibre can be expressed as follows:

$$S = S(\varepsilon) = k\bar{S}(\varepsilon) = [P/\bar{S}(a)]\bar{S}(\varepsilon), \text{ when } \varepsilon \le a \tag{10a}$$

$$S = 0, \text{ when } \varepsilon > a \tag{10b}$$

The average force per fibre in the fibre bundle S^* is given by

$$S^* = \int_{a_{\min}}^{a_{\max}} \int_{P_{\min}}^{P_{\max}} Su(P,a)\,\mathrm{d}P\mathrm{d}a \tag{11}$$

On the basis of the above two assumptions, S^* takes the following forms.

Case 1 (no fibre is broken): substituting S from Equation (10a) into (11) and then utilising (7), we obtain

$$S^* = \bar{S}(\varepsilon)\int_{a_{\min}}^{a_{\max}} \frac{\overline{P(a)}}{\bar{S}(a)} g(a)\,\mathrm{d}a, \quad \text{when } \varepsilon < a_{\min} \tag{12a}$$

Case 2 (fibres with $a < \varepsilon$ are broken): in analogy to the derivation of Equation (12a), we obtain

$$S^* = \bar{S}(\varepsilon)\int_{\varepsilon}^{a_{\max}} \frac{\overline{P(a)}}{\bar{S}(a)} g(a)\,\mathrm{d}a, \quad \text{when } \varepsilon \in (a_{\min}, a_{\max}) \tag{12b}$$

Case 3 (all fibres are broken): then $S = 0$, hence obviously

$$S^* = 0, \text{ when } \varepsilon > a_{\max} \tag{12c}$$

It is also possible to derive an expression for the breaking force of the fibre bundle related to one fibre. This is the maximum of the average force per fibre in the fibre bundle. In this context, we consider the most common type of force-strain behaviour of a fibre bundle, as shown in Figure 3, with the breaking force of the bundle related to one fibre P^* and the breaking strain of the bundle related to one fibre $\alpha^* \in \langle a_{\min}, a_{\max}\rangle$. Utilising the condition of breakage $(\mathrm{d}S^*/\mathrm{d}\varepsilon)_{\varepsilon=a^*} = 0$ of the fibre bundle on Equation (12b) and then rearranging it, we obtain:

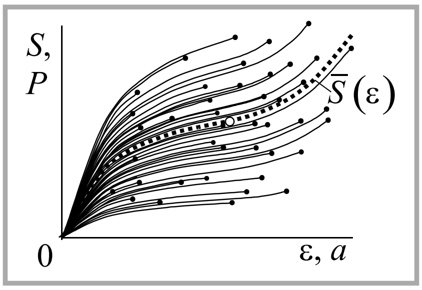

Figure 2. *Fibre tensile curves and concept of similar force-stress relation in fibres.*

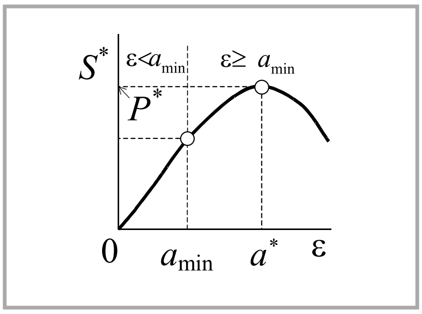

Figure 3. *Most common type of force-strain curve of a fibre bundle.*

$$\frac{\dfrac{\mathrm{d}\bar{S}(a^*)}{\mathrm{d}a^*}\displaystyle\int_{a^*}^{a_{\max}} \frac{\overline{P(a)}}{\bar{S}(a)} g(a)\,\mathrm{d}a}{\overline{P(a^*)}\; g(a^*)} = 1 \tag{13a}$$

Using the symbol corresponding to the breakage of the bundle, that is $\varepsilon = a^*$, in Equation (12b), we obtain:

$$P^* = \bar{S}(a^*)\int_{a^*}^{a_{\max}} \frac{\overline{P(a)}}{\bar{S}(a)} g(a)\,\mathrm{d}a \tag{13b}$$

The roots of Equations (13a) and (13b) are the values of a^* and P^*, respectively.

Assumption of symmetry in breaking forces of fibres

We assume that the conditional average fibre breaking force at a given fibre breaking strain $\overline{P(a)}$ is equal to the corresponding value obtained from the average function $\bar{S}(a)$. Symbolically, $\overline{P(a)} = \bar{S}(a)$. We call this an assumption of symmetry in the breaking forces of fibres. This is schematically shown in Figure 4. Under this assumption, Equations (12a) - (12c) take the following forms:

$$S^* = \bar{S}(\varepsilon), \text{ when } \varepsilon < a_{\min}, \tag{14a}$$

$$S^* = \bar{S}(\varepsilon)[1 - G(\varepsilon)], \text{ when } \varepsilon \in \langle a_{\min}, a_{\max}\rangle, \tag{14b}$$

$$S^* = 0, \text{ when } \varepsilon > a_{\max}; \tag{14c}$$

Thus, (13a) and (13b) can be expressed as follows:

$$\frac{\dfrac{\mathrm{d}\bar{S}(a^*)}{\mathrm{d}a^*}}{\bar{S}(a^*)}\frac{[1-G(a^*)]}{g(a^*)} = 1 \tag{15a}$$

$$P^* = \bar{S}(a)[1 - G(a^*)]. \tag{15b}$$

The roots of Equations (15a) and (15b) are the respective values of a^* and P^* under the assumption of symmetry in breaking force of fibres.

Some relative variables and their uses

We define the relative fibre breaking force y as a ratio of the fibre breaking force $\bar{P}$ to the average fibre breaking force P. Symbolically, $y = P/\bar{P}$. So, $\mathrm{d}y = \mathrm{d}P/\bar{P}$. Analogically, the relative fibre breaking strain z is defined as the ratio between the fibre breaking strain a and the average fibre breaking strain $\bar{a}$. Symbolically, $z = a/\bar{a}$. So, $\mathrm{d}z = \mathrm{d}a/\bar{a}$. We also define the breaking force utilisation coefficient η_P as a ratio between the breaking force of the fibre bundle related to one fibre P^* and the average fibre breaking force $\bar{P}$. Symbolically, $\eta_P = P^*/\bar{P}$. Analogically, the breaking strain utilisation coefficient η_a is defined as a ratio between the breaking strain of the fibre bundle related to one fibre a^* and the average fibre breaking strain $\bar{a}$. Symbolically, $\eta_a = a^*/\bar{a}$.

The distribution of the relative fibre breaking points (y, z) is given by the probability density function $w(y, z)$. From the theory of probability, we obtain

$$w(y, z)\mathrm{d}y\mathrm{d}z = u(P, a)\mathrm{d}P\mathrm{d}a. \tag{16}$$

Then, following the above symbolism, Equation (16) takes the following form:

$$w(y, z) = \bar{P}\bar{a}\, u(P, a). \tag{17}$$

Using Equations (17) and (3) and also the definition of y, the marginal probability density function of the relative fibre breaking strain $h(z)$ can be expressed as

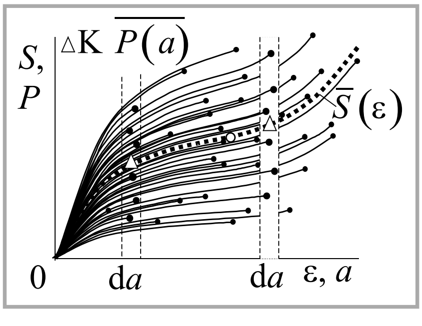

Figure 4. *Concept of symmetry in breaking forces of fibres.*

$$h(z) = \bar{a}\, g(a). \quad (18)$$

From the definitions of z and η_a obviously when $a = a^*$ then $z = \eta_a$. Hence Equation (18) can be written as

$$h(\eta_a) = \bar{a}\, g(a^*). \quad (19)$$

Substituting Equations (17) and (18) in the definition of the conditional probability density function of the relative fibre breaking force at a given relative fibre breaking strain $\varphi(y|z)$, and then comparing the resultant expression with (6), we obtain

$$\varphi(y|z) = \bar{P}\psi(P|a) \quad (20)$$

Substituting Equation (20) and using the definition of y in the definition of the conditional average relative fibre breaking force at a given relative fibre breaking strain $\overline{y(z)}$, we obtain

$$\overline{y(z)} = \overline{P(a)}/\bar{P}. \quad (21)$$

It is already known that when $a = a^*$ then $z = \eta_a$. Hence Equation (21) takes the following form:

$$\overline{y(\eta_a)} = \overline{P(a^*)}/\bar{P}. \quad (22)$$

Now we define the relative fibre strain t as a ratio between the fibre strain e and the average fibre breaking strain $\bar{a}$. Symbolically, $t = \varepsilon/\bar{a}$. So, $dt = d\varepsilon/\bar{a}$. Evidently, this is the relative strain of the fibre bundle also. Under this symbol, we can consider the average function $\bar{S}(\varepsilon)$ as shown below:

$$\bar{S}(\varepsilon) = \bar{P}\zeta(t), \quad (23)$$

where $\zeta(t) = 1/\bar{P}\ \bar{S}(\bar{a}(\varepsilon/\bar{a}))$,

where we call $\zeta(t)$ as the relative average function. From the definitions of t and z, it is obvious that when $\varepsilon = a$ then $t = z$, Equation (23) can thus be expressed as

$$\bar{S}(\alpha) = \bar{P}\zeta(t), \quad (24)$$

From the definitions of t and η_a, it is also obvious that when $\varepsilon = a^*$ then $t = \eta_a$, and so Equation (23) can be expressed in another form, as follows:

$$\bar{S}(a^*) = \bar{P}\zeta(\eta_a), \quad (25)$$

Now the following derivation is evident from Equation (23)

$$d\bar{S}(\varepsilon)/d\varepsilon = (\bar{P}/\bar{a})(d\zeta(t)/dt). \quad (26)$$

It is already known that when $\varepsilon = a^*$ then $t = \eta_a$, and so it is valid to write Equation (26) as

$$(d\bar{S}(a^*)/da^*) = (\bar{P}/\bar{a})(d\zeta(\eta_a)/d\eta_a). \quad (27)$$

Now we define the relative average force per fibre in the bundle σ as $\sigma = S^*/\bar{P}$. This takes the following forms under the three cases mentioned below:

Case 1 (no fibre is broken): From the definitions of t and z, it is obvious that when $\varepsilon < a_{min}$ then $t < z_{min}$. At first, substituting S^* from Equation (12a) into the definition of σ, then utilising (23), (21), (24), (18), and the definition of z, we obtain

$$\sigma = \xi(t)\int_{z_{min}}^{z_{max}}\left(\overline{y(z)}/\xi(z)\right)h(z)dz \quad (28a)$$

Case 2 (fibres with $a < \varepsilon$ are broken): From the definitions of t and z, it is obvious that when $\varepsilon \in \langle a_{min}, a_{max}\rangle$ then $t \in \langle z_{min}, z_{max}\rangle$. In analogy to the derivation of Equation (28a), we obtain

$$\sigma = \xi(t)\int_{t}^{z_{max}}\left(\overline{y(z)}/\xi(z)\right)h(z)dz \quad (28b)$$

Case 3 (all fibres are broken): Obviously, from the definitions of t and z, when $\varepsilon > a_{max}$ then $t > z_{max}$. Under this case, $S^* = 0$, hence obviously

$$\sigma = 0 \quad (28c)$$

Now utilising Equations (25), (21), (24), (18), (22), and (19) into (13a) and then utilising the definitions of z and η_a, we obtain

$$\frac{\dfrac{d\xi(\eta_a)}{d\eta_a}\displaystyle\int_{\eta_a}^{z_{max}}\frac{\overline{y(z)}}{\xi(z)}h(z)dz}{\overline{y(\eta_a)}\,h(\eta_a)} = 1, \quad (29a)$$

$$\text{when } \eta_a \geq z_{min}$$

At first, substituting P^* from Equation (13b) in the definition of η_P and then utilising (25), (21), (24), (18), and the definition of z, we obtain

$$\eta_P = \xi(\eta_a)\int_{\eta_a}^{z_{max}}\frac{\overline{y(z)}}{\xi(z)}h(z)dz \quad (29b)$$

The roots of Equations (29a) and (29b) are the values of η_a and η_P respectively.

Under the assumption of symmetry in breaking forces of fibres, using Equations (21) and (24) the following expression is obtained:

$$\overline{y(z)} = \zeta(z) \quad (30a)$$

Substituting the random variable z by another random variable η_a in Equation (30a), we obtain

$$\overline{y(\eta_a)} = \zeta(\eta_a) \quad (30b)$$

Substituting Equation (30a) into (28a)-(28c) respectively, we obtain

$$\sigma = \zeta(t), \text{ when } t < z_{min}, \quad (31a)$$

$$\sigma = \zeta(t)[1 - H(t)], \text{ when } t \in \langle z_{min}, z_{max}\rangle, \quad (31b)$$

$$\sigma = 0, \text{ when } t > z_{max}; \quad (31c)$$

where

$$H(z) = \int_{z_{min}}^{z} h(z')dz'$$

is the dis tribution function of z. Substituting Equations (30a) and (30b) into (29a), we obtain the following expression:

$$\frac{d\xi(\eta_a)/d\eta_a}{\xi(\eta_a)}\frac{\left[1 - H(\eta_a)\right]}{h(\eta_a)} = 1, \quad (32a)$$

$$\text{when } \eta_a \geq z_{min}$$

Substituting Equation (30a) into (29b), we obtain

$$\eta_P = \zeta(\eta_a)[1 - H(\eta_a)], \quad (32b)$$

Equations (32a) and (32b) allow us to evaluate η_a and η_P respectively under the assumption of symmetry in breaking forces of fibres.

Note: Two ratios are shown at the left-hand side of Equation (32a): the first one represents force-strain relation, and the second one concerns the influence of the distribution of the relative fibre breaking points.

Examples

1) Assume the force-strain relation of fibres is linear. Then the average function must be linear also: $\bar{S}(\varepsilon) = (\bar{P}/\bar{a})\varepsilon$. Comparing this expression with Equation (23) and utilising the definition of t, we obtain $\zeta(t) = t$. So $d\zeta(t)/dt = 1$. Substituting the random variable t by another variable η_a into the relation $\zeta(t) = t$, we obtain $\zeta(\eta_a) = \eta_a$. So $d\zeta(\eta_a)/d\eta_a = 1$.

2) Assume the fibre breaking points (P,a) follow a two-dimensional Gaussian (normal) distribution $u(P,a)$. From the theory of probability, we obtain that the marginal probability density function $g(a)$ of fibre breaking strain must also be Gaussian with average $\bar{a}$ and standard deviation s_a; and the random variable z also follows Gaussian distribution, but with average 1 and standard deviation v_a, where $v_a = s_a/\bar{a}$. Evidently, v_a has the meaning of the coefficient of variation (CV) of the fibre breaking strain. So, the following expressions are valid:

$h(z) = (1/2\pi)\exp[-(z-1)/(2v_a^2)]$ and

$$H(z) = \int_{-\infty}^{z} h(z)\,\mathrm{d}z.$$

Let us now define a standardised random variable u as $u = (z - 1)/v_a$. For $z = \eta_a$, we use the symbol u_a such that $u_a = (\eta_a - 1)/v_a$, and for $z = t$, we use the symbol u_t such that $u_t = (\tau - 1)/v_a$. Clearly, the variable u has the standardised Gaussian probability density function:

$$f(u) = \left(1/\sqrt{2\pi}\right)e^{-u^2/2}$$

and the standardised distribution function:

$$F(u) = \int_{-\infty}^{u} f(u')\,\mathrm{d}u' \ .$$

Comparing the probability characteristics of z and u, we can write $h(z) = f(u)/v_a$ and $H(z) = F(u)$.

3) Assume the symmetry in the breaking forces of the fibres, i.e. $\overline{P(a)} = \bar{S}(a)$.

Under the first and second assumptions, from the two-dimensional Gaussian probability density function $u(P,a)$ of the fibre breaking force P and the fibre breaking strain a, it is possible to derive $\bar{P}/\bar{a} = \rho\ s_P/s_a$, where s_P and s_a are the standard deviations of fibre breaking force and fibre breaking strain respectively, and ρ is the correlation coefficient between the fibre breaking force and the fibre breaking strain. Now the following relations are evident based on the above three assumptions. From Equation (31b), the average force per fibre in the bundle is obtained as $\rho = t[1 - F(1 - 1/v_a)]$. The behaviour of this expression is shown in Figure 5. From Equation (32a), we obtain

$$\frac{1}{u_a v_a + 1}\frac{\left[1 - F(u_a)\right]}{f(u_a)/v_a} = 1.$$

Solving this equation, we obtain u_a, and then the breaking strain utilisation coefficient can be obtained from the earlier expression $\eta_a = u_a v_a + 1$. From Equation (32b), the breaking force utilisation coefficient is obtained as $\eta_P = (u_a v_a + 1)[1 - F(u_a)]$. Evidently, σ, η_a, and η_P depend only on v_a. Suh & Koo [6] experimentally found that the fibre breaking strain as the most significant contributory factor to the bundle tensile properties. The behaviours of η_a and η_P as a function of v_a are shown in Figure 6. Similar results have been found considering the lognormal and Weibull distributions of fibre breaking strain [7].

Blended fibre bundle

Consider a blended fibre bundle consisting of M different components. The partial components are denoted by the serial number $i = 1, 2, ..., m$ as a subscript. Assume each partial component has n_i fibres, and then the total number of fibres in the whole bundle is $n = \sum_{i=1}^{m} n_i$.

The group of n_i fibres of one component can be understood as the i^{th} partial bundle, consisting of fibres of only one component. If we symbolise the average force per fibre of i^{th} partial bundle by S^*_i then the total force on all fibres of the i^{th} partial bundle $S_{\Sigma,i}$ is given by $S_{\Sigma,i} = n_i S^*_i$. Therefore, the resultant force on the whole bundle S_Σ is then $n = \sum_{i=1}^{m} n_i$.

Hence the average force per fibre in the whole bundle S^* is obtained as

$$S^* = S_\Sigma / n = \sum_{i=1}^{m}\left(n_i S^*_i\right) \Big/ n \ .$$

Obviously, the maximum value of force S_Σ is the breaking force of the whole bundle P_Σ, and the strain ε at which the relation $S_\Sigma = P_\Sigma$ holds is the breaking strain of the whole bundle a^*.

Example

Consider a blended fibre bundle consisting of two components ($M = 2$), where the fibres of each component satisfy the following assumptions:

1) Fibre force-strain relations are linear. Then the average function must be linear also: $\bar{S}_i(\varepsilon) = \bar{P}_i/\bar{a})\varepsilon$.

2) The fibre breaking strain follows a Gaussian distribution. Then the probability density function of fibre breaking strain is

$$g_i(a) = \left(1/\sqrt{2\pi}s_{a,i}\right)\exp\left[-\left(a - \bar{a}_i\right)^2 / 2s_{a,i}^2\right]$$

and the corresponding distribution function is given by

$$G_i(a) = \int_{-\infty}^{a} g_i(x)\,\mathrm{d}x\ .$$

3) The fibre breaking forces are symmetrical. Symbolically, $\overline{P_i(a)} = \bar{S}_i(a)$.

Under these assumptions, the average force per fibre of the i^{th} partial bundle S^*_i can be obtained from Equation (15b), as follows:

$$S^*_i = \frac{\bar{P}_i}{\bar{a}_i}\varepsilon\left[1 - G_i(\varepsilon)\right]. \qquad (33)$$

Utilising Equation (33) and the relation $\alpha_i = (n_i/n)(\bar{P}_i/\bar{a}_i)$, where α_i is a characteristic parameter of the respective component, we obtain

$$S^* = (n_1/n)S^*_i + (n_2/n)S^*_i = \\ = \alpha_1\varepsilon[1 - G_1(\varepsilon)] + \alpha_2\varepsilon[1 - G_2(\varepsilon)]. \qquad (34)$$

From reference [8], it is known that $(n_i/n) = q_i(t/t_i)$, where q_i is the mass portion of i^{th} component such that $\sum_{i=1}^{2} q_i = 1$, t_i is the fineness of the i^{th} component and t is the average fibre fineness. We consider another characteristic parameter β_i of the respective component as $\beta_i = (q_i/t_i)(\bar{P}_i/\bar{a}_i)$. Then we obtain

$$S_\Sigma = nS^* = \\ = T\{\beta_1\varepsilon[1 - G_1(\varepsilon)] + \beta_2\varepsilon[1 - G_2(\varepsilon)]\}. \qquad (35)$$

where $T = nt$ is the fineness of the whole bundle. Now applying the condition of stress maximisation $(\mathrm{d}S_\Sigma/\mathrm{d}\varepsilon)_{\varepsilon=a^*} = 0$ or $(\mathrm{d}S^*/\mathrm{d}\varepsilon)_{\varepsilon=a^*} = 0$ on Equation (34), we obtain

$$\frac{1}{a^*}\frac{\beta_1\left[1 - G_1(a^*)\right] + \beta_2\left[1 - G_2(a^*)\right]}{\beta_1 g_1(a^*) + \beta_2 g_2(a^*)} = 1 \qquad (36)$$

The numerical solution of Equation (36) can give one to three roots. The 'correct' root, which corresponds to the actual breaking strain of the whole bundle a^*, is determined from the equation for calculation of breaking force. The breaking strain of the whole bundle a^* and the breaking force of the whole bundle P_Σ are the coordinate of one point that lies

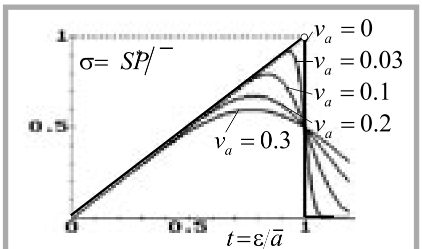

Figure 5. *Average force per fibre in the bundle vs. Relative fibre strain at different CV of fibre breaking strain.*

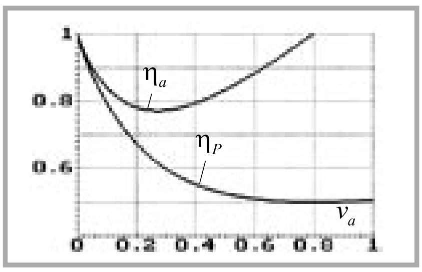

Figure 6. *Breaking force and breaking strain utilisation coefficients versus CV of fibre breaking strain.*

on the force-strain curve expressed by Equation (35). Therefore, we can write

$$P_{\Sigma} = T\{\beta_1\varepsilon[1 - G_1(\varepsilon)] + \beta_2\varepsilon[1 - G_2(\varepsilon)]\} \quad (37)$$

If Equation (36) has more roots, then the root leading to the highest value of P_{Σ} found from Equation (37) is the required breaking strain of the whole bundle a^*. Evidently, from Equation (37), it is possible to obtain the breaking tenacity of the whole bundle $p_{\Sigma} = P_{\Sigma}/T$.

The above theory is illustrated with the help of two imaginary blended fibre bundles (FB 1 and FB 2), where each bundle consists of two different components. The fibres of each component have the following characteristics, as shown in Table 1. (In FB 1, component 1 is like polyester and component 2 is like cotton.) Figure 7a represents the tenacity-strain curves of FB 1 obtained from Equation (35) using the expressions for β_i, $g_i(a)$, $G_i(a)$ as considered before and the relation $g_1 + g_2 = 1$. The curves are almost bimodal, except for bundles with only one component, i.e., $g_1 = 0$ or $g_1 = 1$. Figure 7b illustrates the tenacity-strain curves of FB 1 on the basis of Hamburger's theory ($s_{a,1} \rightarrow 0$ and $s_{a,2} \rightarrow 0$) [3]. The effect of variability in the breaking strain of fibres within a component on the force-strain behaviour of FB 1 can be understood by comparing these both sets of curves. By solving Equation (36) using the expressions for β_i, $g_i(a)$, $G_i(a)$ as considered before and the relation $g_1 + g_2 = 1$, we obtain the thick lines in Figures 8a and 8b showing the effect of blend ratio on the breaking tenacity and breaking strain of FB 1, respectively. The thin lines in Figures 8a and 8b are obtained on the basis of Hamburger's theory ($s_{a,1} \rightarrow 0$ and $s_{a,2} \rightarrow 0$) [3]. Evidently, the shifting of the thick and thin lines is significant. (It is not true that all the fibres of one component break at the same time.) In the case of FB 2, where the fibres of one component differ from the other component only in terms of variability in fibre breaking strain, the effect of blend ratio on the bundle breaking tenacity and breaking strain is shown in Figure 9. Evidently, the change of shape and the shifting of the thick and thin lines are significant. (The overlapping of the distributions of fibre breaking strain of the components is significant.)

Table 1. *Characteristics of fibres in bundles FB 1 and FB 2.*

Fibre parameters	FB 1		FB 2	
	Component 1	Component 2	Component 1	Component 2
Average breaking tenacity p_i, N/tex	0,5	0,3	0,3	0,3
Average breaking strain a_i, %	30	8	16	8
Standard deviation of breaking strain $s_{a,i}$	0,015	0,024	0,032	0,032
CV of breaking strain $v_{a,i}$, %	5	30	20	40

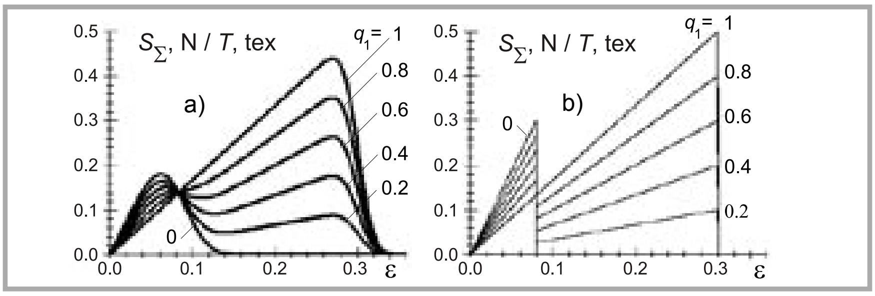

Figure 7. *Tensile curves (tenacity-strain) of the fibre bundle FB 1; a) the presented theory b) Hamburger's theory.*

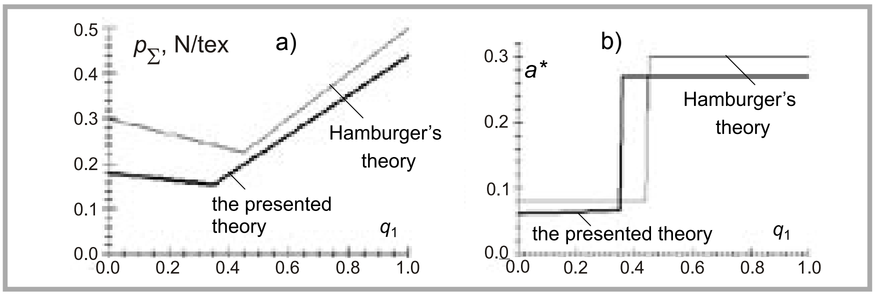

Figure 8. *Comparison between the presented theory and Hamburger's theory with a view to the tensile behaviours of fibre bundle FB 1; a) Breaking tenacity vs. mass portion b) Breaking strain vs. mass portion.*

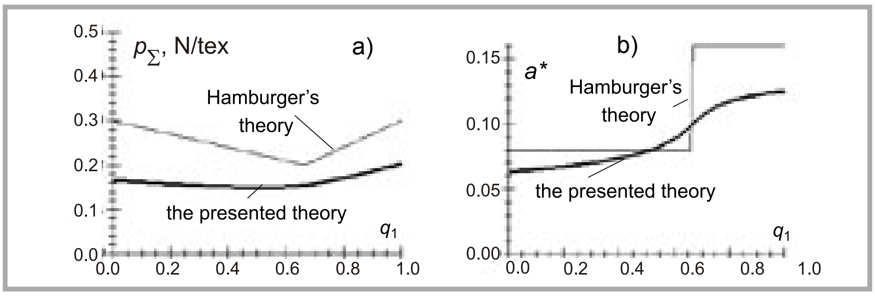

Figure 9. *Comparison between the presented theory and Hamburger's theory with a view to the tensile behaviours of fibre bundle FB 2; a) Breaking tenacity vs. mass portion b) Breaking strain vs. mass portion.*

Conclusion

This work shows that it is possible to model the tensile behaviour of fibre bundles, where the constituent fibres possess different tensile behaviours. Extrapolating this fact into the model proves to be significant when predicting the tensile behaviour of the bundle; this behaviour is found to be different than that obtained from Hamburger's theory. It is shown that the average force per fibre in the bundle, the breaking force utilisation coefficient, and the breaking strain utilisation coefficient depend only on the coefficient of variation of fibre breaking strain. It will be very useful to produce a set of blended fibre bundles and yarns under comparable parameters (material, technology, etc.), and experimentally verify the above theoretical model. Working out supplementary empirical corrections to this model will lead to a practical way for predicting the tensile behaviour of blended fibre bundles and yarns.

List of symbols

P	*Fibre breaking force*
a	*Fibre breaking strain*
$u(P,a)$	*Joint probability density function of fibre breaking force and fibre breaking strain*
P_{min}	*Minimum fibre breaking force*
P_{max}	*Maximum fibre breaking force*
a_{min}	*Minimum fibre breaking strain*
a_{max}	*Maximum fibre breaking strain*
$\bar{P}$	*Average fibre breaking force*
$\bar{a}$	*Average fibre breaking strain*
$g(a)$	*Marginal probability density function of fibre breaking strain*
$G(a)$	*Distribution function of fibre breaking strain*
$\psi(P\|a)$	*Conditional probability density function of fibre breaking force at a given fibre breaking strain*
$\overline{P(a)}$	*Conditional average fibre breaking force at a given fibre breaking strain*
S	*Force on a fibre*
ε	*Strain on a fibre*
$S(\varepsilon)$	*Force on a fibre at a given fibre strain*
$\bar{S}(\varepsilon)$	*Average force on fibres at a given fibre strain*
k	*Fibre parameter*
$S(a)$	*Force on a fibre at a given fibre breaking strain*
$\bar{S}(a)$	*Average force on fibres at a given fibre breaking strain*
$\bar{S}(\bar{a})$	*Average force on fibres at a given average fibre breaking strain*
S^*	*Average force per fibre in a fibre bundle*
P^*	*Breaking force of a fibre bundle related to one fibre*
a^*	*Breaking strain of a fibre bundle related to one fibre*
$\bar{S}(a^*)$	*Average force on fibres at a given breaking strain of a fibre bundle related to one fibre*
$G(\varepsilon)$	*Distribution function of strain on fibres*
$G(a^*)$	*Distribution function of breaking strain of a fibre bundle related to one fibre*
$g(a^*)$	*Marginal probability density function of breaking strain of a fibre bundle related to one fibre*
y	*Relative fibre breaking force*
z	*Relative fibre breaking strain*
η_P	*Fibre breaking force utilisation coefficient*
η_a	*Fibre braking strain utilisation coefficient*
$w(y,z)$	*Probability density function of relative fibre breaking force and relative fibre breaking strain*
$h(z)$	*Marginal probability density function of relative fibre breaking strain*
$h(\eta_a)$	*Marginal probability density function of fibre breaking strain utilisation coefficient*
$\varphi(y\|z)$	*Conditional probability density function of relative fibre breaking force at a given relative fibre breaking strain*
$\overline{y(z)}$	*Conditional average relative fibre breaking force at a given relative fibre breaking strain*
$\overline{y(\eta_a)}$	*Conditional average relative fibre breaking force at a given fibre breaking strain utilisation coefficient*
$\overline{P(a^*)}$	*Conditional average fibre breaking force at a given breaking strain of a fibre bundle related to one fibre*
t	*Relative fibre strain*
$\zeta(t)$	*Relative average function of fibre strain*
$\zeta(z)$	*Relative average function of fibre breaking strain*
$\zeta(\eta_a)$	*Relative average function*
σ	*Relative average force per fiber in a fiber bundle*
z_{min}	*Minimum relative fiber breaking strain*
z_{max}	*Maximum relative fiber breaking strain*
$H(t)$	*Distribution function of relative fiber strain*
$H(z)$	*Distribution function of relative fiber breaking strain*
$H(\eta_a)$	*Distribution function of fiber breaking strain utilization coefficient*
s_a	*Standard deviation of fiber breaking strain*
v_a	*Coefficient of variation of fiber breaking strain*
u,u_a,u_t	*Standardized random variables*
$f(u)$	*Guassian probability density function of u*
$F(u)$	*Distribution function of u*
s_p	*Standard deviation of fiber breaking force*
ρ	*Correlation coefficient between fiber breaking force and fiber breaking strain*
M	*No. of components (partial bundles) present in a blended fiber bundle*
i	*Serial number denoting partial bundle, i = 1, 2, ..., m*
n_i	*No. of fibers present in i^{th} partial bundle*
n	*Total no. of fibers present in a fiber bundle*
S_i^*	*Average force per fiber of i^{th} partial bundle*
$S_{\Sigma,i}$	*Total force on all fibers of i^{th} partial bundle*
S_Σ	*Total force on a fiber bundle*
P_Σ	*Breaking force of a fiber bundle*
$\bar{S}_i(\varepsilon)$	*Average force on fibers of i^{th} partial bundle at a given fiber strain*
$\bar{P}_i$	*Average breaking force of fibers of i^{th} partial bundle*
$\bar{a}_i$	*Average breaking strain of fibers of i^{th} partial bundle*
$g_i(a)$	*Marginal probability density function of breaking strain of fibers of i^{th} partial bundle*
$s_{a,i}$	*Standard deviation of breaking strain of fibers of i^{th} partial bundle*
$G_i(a)$	*Distribution function of breaking strain of fibers of i^{th} partial bundle*
$\overline{P_i(a)}$	*Conditional average breaking force of fibers of i^{th} partial bundle at a given fiber breaking strain*
$\bar{S}_i(a)$	*Average force on fibers of i^{th} partial bundle at a given average fiber breaking strain*
$G_i(\varepsilon)$	*Distribution function of strain on fibers of i^{th} partial bundle*
α_i, β_i	*Parameters characteristic to i^{th} component*
q_i	*Mass portion of i^{th} component*
t_i	*Fineness of fibers of i^{th} component*
t	*Average fiber fineness*
T	*Fineness of the fiber bundle*
$g_i(a^*)$	*Marginal probability density function of breaking strain of fibers of partial bundle*
p_Σ	*Breaking tenacity of a fiber bundle*

Acknowledgment

This research work was supported by the Research Centre 'Textile', No. 1M4674788501.

References

1. A. A. Sinitsin, The Designing of Yarn and Fabric with Prediction of Breaking Loads (in Russian), Gizlegprom, 1932.
2. H. A. Hancock, Egyptian Cotton: Studies in Spinning and Growing, Egypt, 1951.
3. W. J. Hamburger, Journal of Textile Institute, 1949, vol. 40, pp. P700-P718.
4. W. Żurek, The Structure of Yarn, Foreign Scientific Publications Department of the National Center for Scientific, Technical, and Economic Information, Poland, 1971, pp. 325-327.
5. A. Kemp and J. D. Owen, Journal of Textile Institute, 1955, vol. 46, p. T648.
6. M. W. Suh and H. J. Koo, Estimation of Bundle Modulus and Toughness from HVI Tenacity-Elongation Curves, A Presentation to the EFS System Research Forum, North Carolina, November 6-7, 1997.
7. B. Neckář, Morphology and Structural Mechanics of General Fibre Assemblies (in Czech), Technical University of Liberec, Czech Republic, 1998, pp. 137-165.
8. B. Neckář and S. Ibrahim, Structural Theory of Fibrous Assemblies and Yarns, Part I, Technical University of Liberec, Czech Republic, 2003, p.21.

Received 07.08.2004 Reviewed 20.04.2006