Vytautas Milašius, Dalia Neverauskienė, Jurgis Katunskis, Ilona Kazlauskienė

Kaunas University of Technology Department of Textile Technology,

Studentu 56, Kaunas, LT 3031, Lithuania,
Tel: +3707 451645; Fax: +3707 353989
E-mail: Vytautas.Milasius@dtf.ktu.lt

The Mathematical Basis of Ornamentation of Patterned Woven Fabrics

Abstract

The development of the method presented was made considering the mathematical basis of the ornamentation and technological features of the fabrics. Building on the works of H. J. Woods, M. A. Hann, and using original investigations, a method for analysis and synthesis of ornaments within the context of CAD for woven textile design is proposed. Using this method, the new computerised creation of woven ornaments based on matrix multiplication is proposed for analysis and synthesis of the ornaments of woven fabrics. We discuss some limitations which exist in the creation of fabric ornamentation when the ornament is obtained in the weaving process.

Key words: *woven ornament, segment, symmetry operations, symmetry group, computerised creation of ornaments.*

Introduction

The aesthetic appearance of woven fabric is one of the oldest and most important of its functions. For this reason, woven fabrics were (and are) often decorated by various ornaments. Among the technologies - printing, embroidery, weaving - the latter one is the most complicated and thus the most valuable way of decoration. The ornament is the main element of patterned fabric design. The patterned woven fabrics have had an important place in the customs and traditions of various nations. They often had not only material significance, but symbolic meaning as well. In Lithuania, J. Balčikonis [1], A. Tamošaitis [2,3], V. Tuménas [4], V. Savoniakaité [5] and others have developed the ethnographic and artistic analyses of the textile inheritance. On the other hand, it is quite clear that analysis based on a mathematical and technological approach is also very important and significant [6], because there is no other area of folk art which was influenced as much by technology in the stylisation form of its symbols as woven textiles. While analysing the ornamentation it is noted that ornaments conform to symmetry concepts, a perspective which has its origins in the scientific investigation of crystals. The English physicist H. J. Woods has proposed a general system of notation and classification of ornaments [7], which was later improved by M. A. Hann et al. [8,9]. This system covers the ornaments that can be created by all possible technology. It is expected that woven ornaments also conform to the general symmetry concepts, and the general system of notation and classification by Woods-Hann is applicable for this kind of ornaments. On the other hand, woven designs have some specific limitations caused by technology and the structure of the woven fabric. Recent research gives the main attention to the ornamentation of patterned fabrics from the standpoint of technology as well as fundamental geometric rules.

Aim and Methods of Investigation

There are four structures of the woven fabrics usually used for creation of the patterned dobby fabrics - broche figured cloth, fabrics with extra warp threads, fabrics with extra weft threads, and block weave fabrics. The aim of this investigation is the creation of a computerised system designed for all these types of woven fabric ornaments.

The research method chosen in this work is based on the system approach - the decomposition of ornamentation into structural components, description by parameters, and evaluation of interaction. In the mid-1930s, H.J. Woods [7] published a number of papers in which he proposed connecting mathematical rules with the geometrical structures of ornaments in two dimensions. These rules are based on four symmetry operations. Any ornament can be created according to these operations:

- translation - the basic ornamentation segment is repeated at regular intervals and in any single direction, when the same orientation of segment is retained;
- rotation - the basic ornamentation segment is repeated at regular angular intervals around an imaginary fixed point called the centre of rotation;
- reflection - there is a mirror image of the basic ornamental segment across an imaginary line called the reflection axis;
- glide-reflection - there is a combination of translation and reflection operations in association with the glide-reflection axis.

Woods' classification and notation system of ornamentation (later improved by Hann [8,9]) divides all regularly repeating ornaments into three classes: finite ornaments, monotranslational ornaments and ditranslational ornaments. These three classes, depending on the combination of symmetry operations and the lie of the centres and axes of symmetry, are respectively subdivided into two, seven and seventeen groups.

The notation of ornament consists of four symbols that identify the constituent symmetrical characteristics associated with each symmetry group.

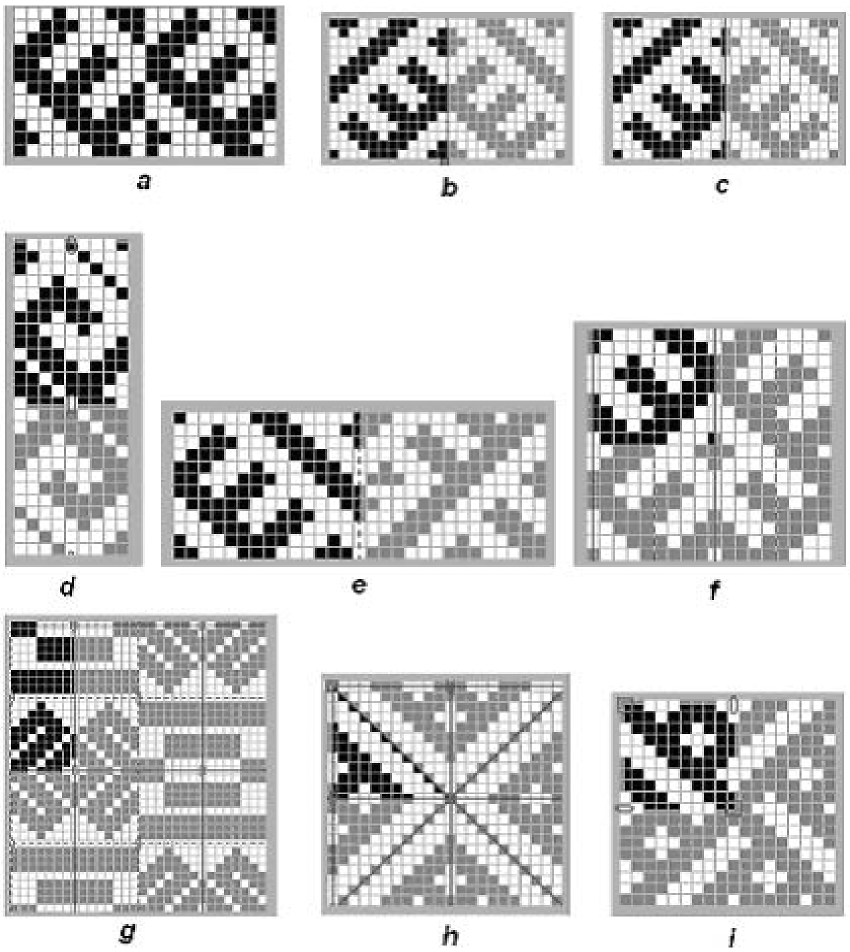


Figure 1. Forming of the woven ornament: a - using translation operation, b, c - using reflection operation, d - using rotation operation, e - using glide-reflection operation, f - with centred cell of c1m1 symmetry group, g - with centred cell of c2mm symmetry group, h - using symmetry group with four-fold rotation p4mm, i - using symmetry group with four-fold rotation p411.

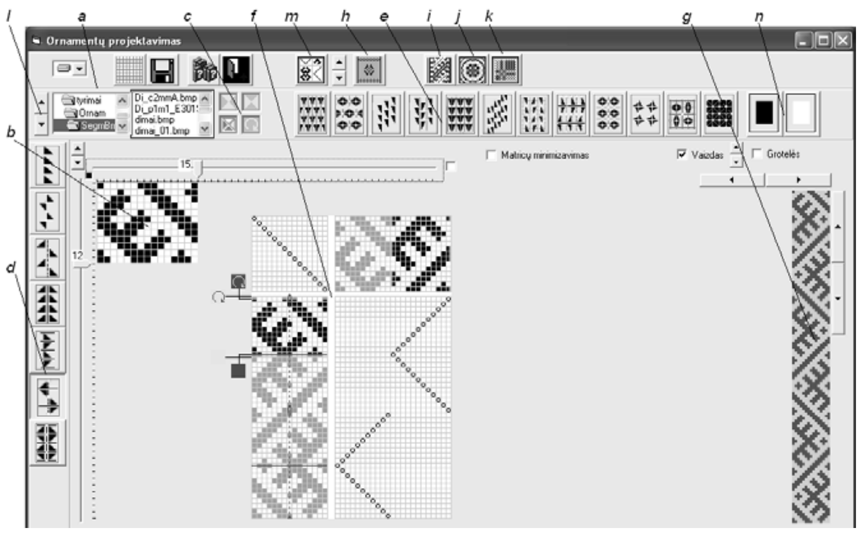


Figure 2. Fragment of the main window of the program for creating the woven ornaments: a - part of the window for the segment creation - design or choice from catalogue, b - view of the segment, c - elements for segment modification (for example, its rotation, horizontal reflection, and so on), d - one of the main toolbars subservient to design monotranslation symmetry groups, e - the second of the main toolbars subservient to design ditranslational symmetry groups, f - view of one ornaments' repeat in the ornament field (the ornament designed can be presented as the ornament itself or in an expanded form view), g - the view of the fabric, h - a button subordinated to create the weave plan of the broche-type fabrics, i - creation of the weave plan for the fabrics with extra warp threads, j - button for the fabrics with extra weft threads, k - button for block weave fabrics, l - a number of buttons subordinated to the zoom operations.

The first symbol of monotranslational motifs is common to all symmetry groups (*p*). The second symbol defines whether vertical reflection axes are presented (*m*) or not (*1*). The third symbol defines the features of the horizontal axis: either it is not presented (*1*) or it is presented; in the last case, either reflection axis (*m*) or glide-reflection axis (*a*) is presented. The fourth symbol defines whether two-fold rotation is presented (*2*) or not (*1*). More explanation about the Woods-Hann system is given in [7-9].

The Application of the Woods-Hann System to Woven Ornaments

Some limitations exist in the creation of fabric ornamentation when the ornament is created in the weaving process. Firstly, not all symmetry groups of the Woods-Hann system can be used. Only twelve of all seventeen ditranslational symmetry groups can be used for the description of woven ornaments. Five remaining groups, in which three-fold (*p311*, *p3m1*, *p31m*) or six-fold (*p611*, *p6mm*) rotation is used, cannot be used in this case because the ornamental motifs of these groups cannot be delineated rectangularly, which is necessitate for the structure of the fabric's weave repeat.

The other limitations appear because the position of axes or the centres of symmetry operations in the basic segment of the woven ornament can be laid between columns (rows) or on the column (row) conditionally called as threads. Limitations come into being only when axes or centres are laid on the threads. The limitations of every symmetry operation are given below.

In the translation operation, the basic segment is only repeated, so this symmetry operation has no limitation (Figure 1a). In the reflection operation, the basic segment is reflected across an axis, and this axis can be laid either on or between threads. This symmetry operation can be used with any segment. If the reflection axis is laid between threads, the last warp of the segment is repeated (Figure 1b). If the axis is laid on the thread, it divides this thread in two parts, but a whole thread is formed when the reflection operation is accomplished (Figure 1c).

In the rotation operation, any segment can be used too, if the centre of rotation is laid conditionally between the threads. Otherwise, when the centre is laid on the thread (Figure 1d), this thread must be symmetrical (marked squares and blank squares must be laid symmetrically). If the rotation is performed on the symmetrical thread, the half thread, formed in this operation will supplement the leftover part of the thread (these parts are equal) and the whole thread is formed in this way. But if the segment contravenes these conditions, the rotation operation cannot be used, because rotating the half thread without the supplement of the leftover thread part is impossible.

Glide-reflection, when the axis is laid on the thread, is the next symmetry operation limited by special conditions (Figure 1e). Control of the thread on which the glide-reflection axis is laid is necessary. During this operation, the segment is reflected and later translated by a certain step of the shift. In this case the glide-reflection axis divides the thread into two parts. It is thus necessary to choose the step of translation which would ensure formation of the whole parts of the ornament. The algorithm of the translation step's establishment was proposed.

First of all, the number of the squares of segment thread on which the glide-reflection axis is laid is divided by each natural number within the interval [*2; R/2*] (*R* - the number of the thread's squares). Only whole numbers from all quotients need be chosen and verified. The thread is divided into segments, whose lengths are equal to the quotient. These segments have to be equal to each other; their layout of marked and blank squares has to be identical. The verified quotient is saved if it obeys the above-mentioned conditions. If such quotients are not found, the glide-reflection operation is impossible for the same reason as the rotation operation. The lowest number (*k*) must be chosen from all the saved quotients, and the shift of glide-reflection (or possible shifts of this transformation) can be found by this formula:

$$p=nk \qquad (1)$$

where:
p - step of the transformation,
n - the whole number from interval [*1; R/k - 1*].

Thus, not every segment can be used for glide-reflection operation if the axis of this operation is laid on the tread. If the chosen step of translation contravenes above-mentioned conditions, the glide-reflection operation is impossible. In Figure 1e, the glide-reflection symmetry whose axis is laid on the thread is shown. A glide-reflection operation for this segment is possible, and the translation' step can only be 6. There are no more limitations in the projection of monotranslational ornaments.

Table 1. *Examples of the Woods-Hann code, scheme, and new ornament presentation of some monotranslational symmetry groups.*

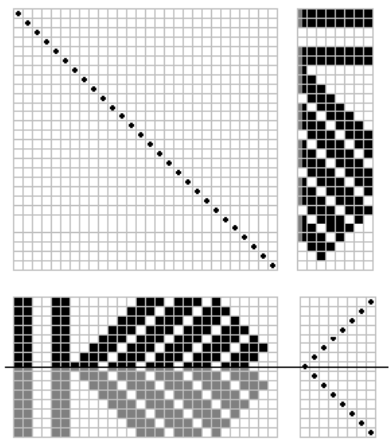
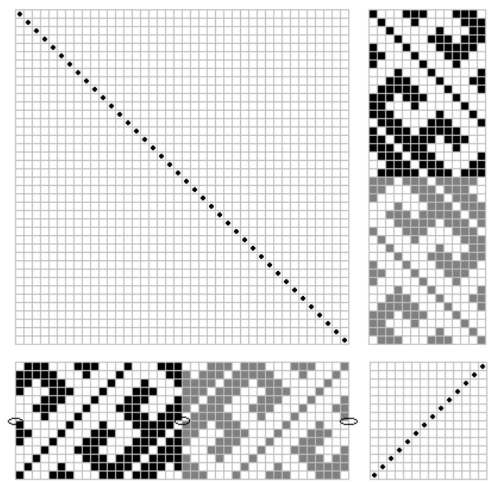
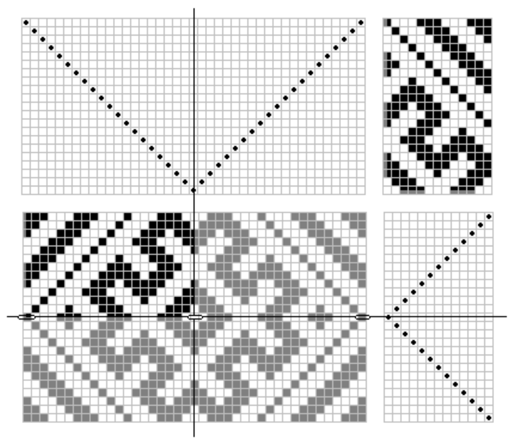
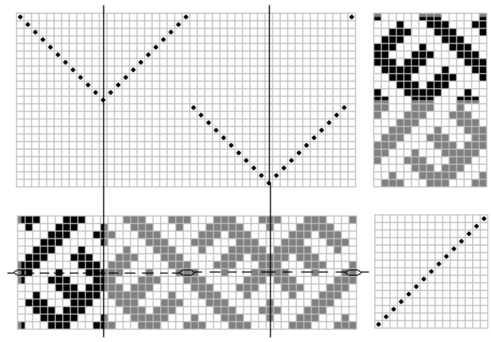
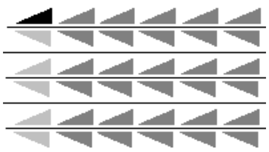
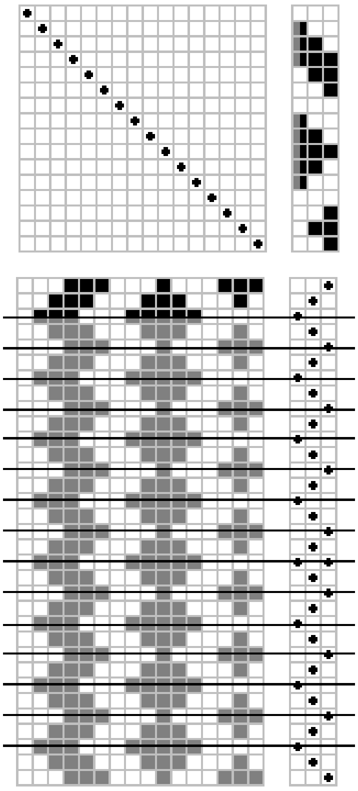
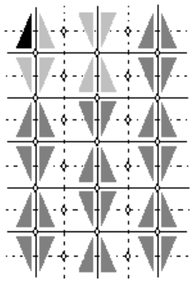
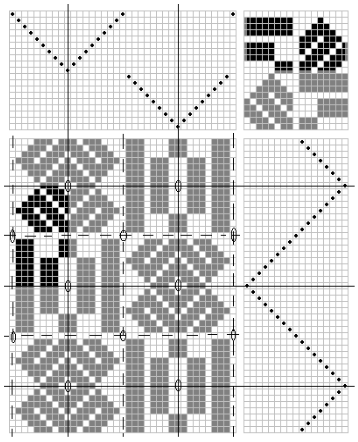
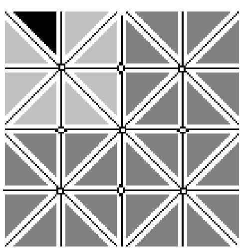
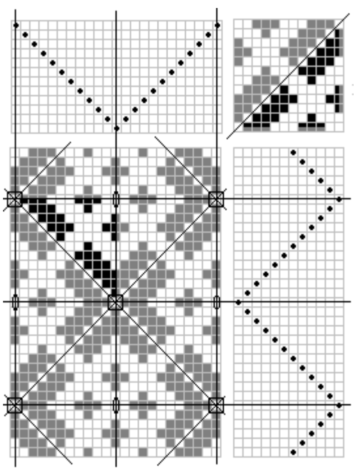
Scheme/Code	Ornament
 p1m1	
 p112	
 pmm2	
 pma2	

Table 2. *Examples of the Woods-Hann code, scheme, and new ornament presentation of some ditranslational symmetry groups.*

Scheme/Code	Ornament
 *p1m1*	
 c2mm	
 p4mm	

The creation of ditranslational ornaments is limited by the same conditions as that of monotranslational ornaments, and has two extra conditions. One condition is typical of two symmetry groups which have central cells; their formula's first symbol is *c* (*c1m1*, *c2mm*). In these ornamentation groups, the motifs (cells) are laid in centre order, that is to say, the central axis of the first row's motif checks with outside axis of the second row's motif, and *vice versa*. For this reason, the condition arises that all vertical or horizontal axes must be the same (laid on or between threads) (Figure 1f,g). The vertical and horizontal axes of *c2mm* symmetry group can be different. The next condition is typical only for creating ornaments of symmetry groups in which the four-fold rotation is used. In these symmetry groups alone, the basic segment must be an isosceles triangle with a straight angle (*p4mm*) (Figure 1h) or a square (*p411* and *p4gm*) (Figure 1i). Other symmetry groups have rectangular segments. The choice of position of the symmetry axes and centres of rotation is important because the form of the basic segments cannot be changed; otherwise the ornament designed belongs to another symmetry group.

The present analysis is based on separating the smallest basic segment of ornament, determination and description symmetry operations carried out forming the ornamentation, and on further presentation of the ornament by initial matrices. The method of matrix multiplication is proposed for the analysis and composition of the woven ornaments. According to this method, the ornaments were interpreted as 2D matrices, and the two matrices of horizontal and vertical transformation of the initial ornament matrix were obtained. The matrix of horizontal transformation of the ornament is created by a method similar to the threadling of the weave. The matrix of vertical transformation is also created in a similar manner. The third matrix, named as the basic element of the ornament, can then be calculated as follows:

$$BE = \left(HT \times O^T\right) \times VT \qquad (2)$$

where:

BE - matrix of the basic element of the ornament,

HT - matrix of the horizontal transformation order of the ornament,

VT - matrix of the vertical transformation order of the ornament,

O - matrix of the ornament.

So, all the motifs of patterned fabrics can be decompounded into three initial elements - horizontal transformation order, vertical transformation order, and basic element. If these initial elements are known, the matrix of the ornament can be calculated as follows:

$$O = \left(VT \times BE^{T}\right) \times HT \qquad (3)$$

The initial matrices can be shown in a graphic form similar to that used for the plan of the weave. This form allows its use for technological needs, i.e., for loom setting.

Tables 1 and 2 disclose the coherence between the ornaments scheme by Woods-Hann and the initial elements of the proposed ornament. The black elements of the ornament scheme correspond with the black part of the ornaments' draw; the same occurs again with light grey elements. The dark grey elements in the ornament scheme are those which have no corresponding parts in the ornament. All the reflection axes and the centre of two-fold rotation are drawn in both drawings by the same manner. Some examples of the main ornament groups are presented in matrix form in the third column of these tables. Three initial matrices - horizontal transformation order, vertical transformation order, and basic element - enlarge the ornament matrix.

The first ornament created by the code *p1m1* is presented in the first row of Table 1. One can see that, due to the presence of longitudinal reflection axis in the ornament, the matrix of vertical transformation order has the mirror image across the horizontal axis, and the matrix of the basic element fully corresponds to the basic segment of the ornament, rotated clockwise.

The ornament described by code *p112* is presented in the second row of Table 1. The absence of the reflection axes ensures that both matrices of the horizontal and vertical transformation order are simple diagonal matrices, but due to the presence of the two-fold rotational symmetry in the ornament, an extra group of the matrix rows appears in the matrix of the basic element below the segment of the ornament.

The ornament presented in the third row of Table 1 (code *pmm2*) has both transverse and longitudinal reflection axes, and two-fold rotational symmetry. Thus both matrices of horizontal and vertical transformation order have the mirror image across vertical and horizontal axes respectively.

The last ornament of Table 1 is coded as *pma2*. The presence of the trans-

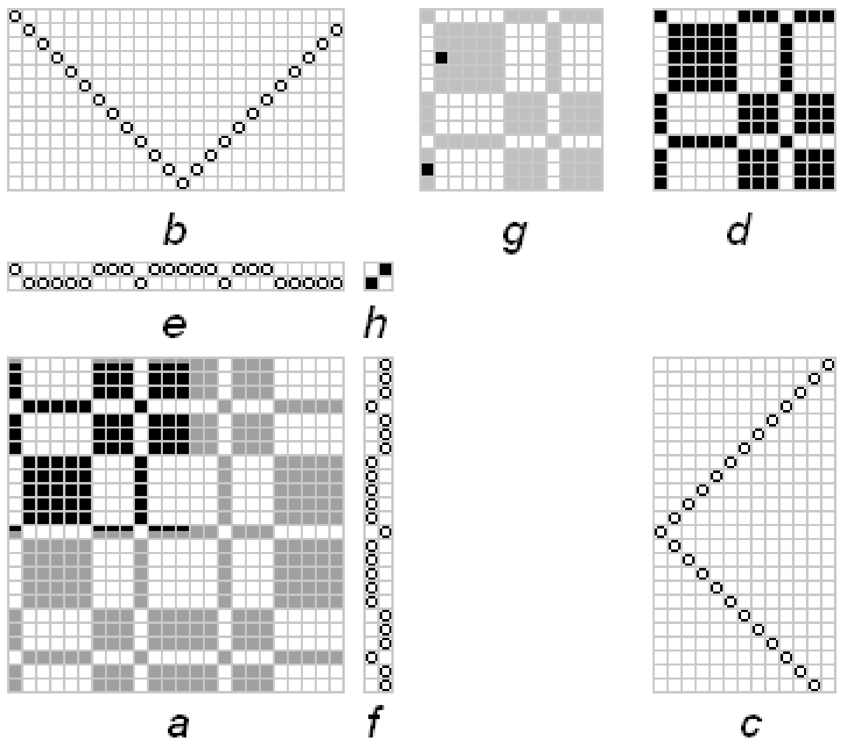


Figure 3. *Transition from the primary form of initial matrices to the squeezed form: a - ornament, b - horizontal transformation order, c - vertical transformation order, d - primary basic element, e - horizontal transformation order treated as if it is treadling, f - vertical transformation order treated as if it is treadling, g - draft of forming the new basic element, h - new basic element.*

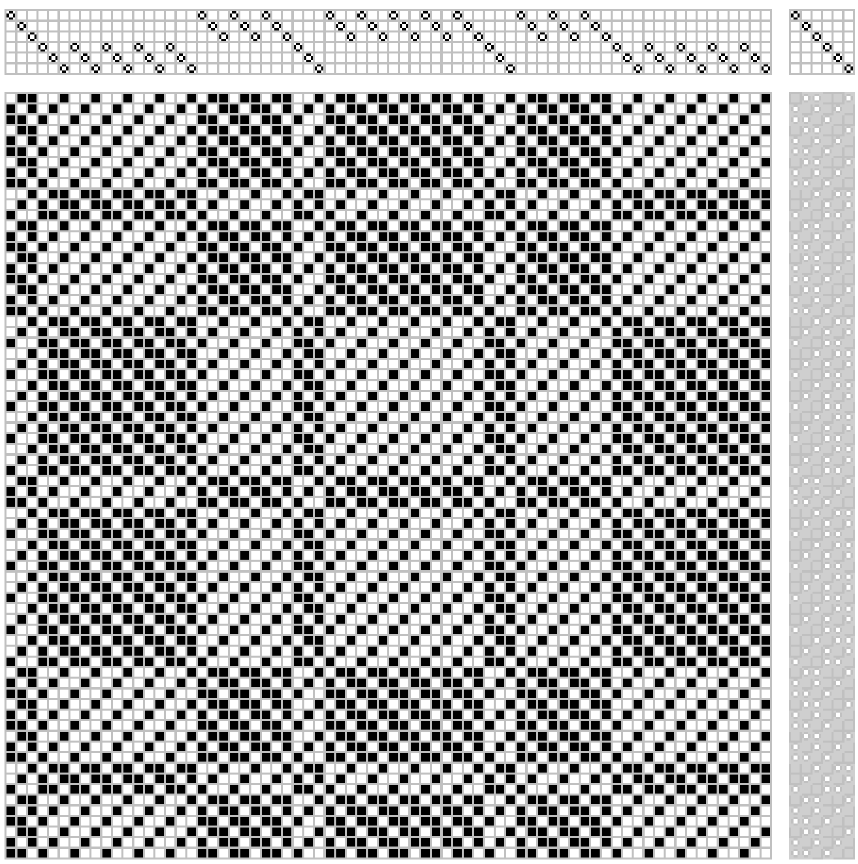


Figure 4. *The weave plan created by CAD designed for block weave fabric according to the ornament from Figure 3.*

verse reflection axis causes the appearance of a mirror image in the matrix of horizontal transformation order. It is also necessary to stress that the presence of the longitudinal glide-reflection axis causes the appearance of the extra rows in the matrix of the horizontal transformation order. For this reason the group of extra rows also appears in the matrix of the basic element. These extra rows of the basic element contain the mirror image of the main segment across the vertical axis.

Some main examples of ditranslational symmetry groups of patterned woven fabrics are presented in Table 2. The simple motif coded as *p1m1* merely has a reflection axis perpendicular to the vertical side of the unit cell, i.e., the horizontal reflection axis. The presentation elements in the matrix form are created in the same way as those of the ornament presented in the first row of Table 1.

The more complex ornament is presented in the second row of Table 2 (code *c2mm*). In this case, both matrices of the horizontal and vertical transformation order have a mirror image (due to the presence of reflection axes) as well as extra row groups (due to the presence of glide-reflection). As ever, the matrix of the basic element contains the segment of the ornament (in black) and an extra part below this segment, which in this case contains the mirror image across the vertical axis of the segment.

The ornament presented in the third row (Table 2) has the code *p4mm*. In this case the four-fold rotational symmetry of the ornament is new. Due to this symmetry, the matrix of the basic element contains the main segment of the ornament and its mirror image across the diagonal reflection axis.

Three initial matrices - horizontal transformation order, vertical transformation order and basic element - can present the ornament's matrix. They are shown in graphic form similar to that used for the plan of the weave. This form allows its use for technological needs, i.e., for loom setting. It is quite clear that this method can be used as a basis for developing the software for the CAD of patterned woven textiles.

Computerised Creation of Woven Ornaments

The creation of this program was based on the Woods-Hann classification and notation system of ornaments and original investigation presented above. This program is especially designed for woven ornaments. The main window of the program is shown in Figure 2.

As shown in Figure 2, the ornament can be presented in expanded form containing all the initial matrices as well as just the ornament itself. The ornament of the patterned fabrics usually is produced with a dobby shedding system (including a hand loom dobby). For this reason, the ornament is treated as if the weave and the matrices of threadling and treadling have been created. Thus, the expanded form (Figure 3a,b,c) must be redrafted to the new compact form, eliminating the rows which represent the repeating columns of drafting and the columns which represent the repeating rows of the treadling (Figure 3e,f). The new basic element must be redrafted eliminating the rows which represent the repeating columns of treadling and the columns which represent the repeating rows of treadling (Figure 1g, grey marks).

This new way of presentating the ornament (Figure 3a,e,f,h) can be used for a technological purpose, namely for the loom setting. The weave plan (Figure 4) can be created according to the structure of the fabric it is designed to produce (broche figured cloth, with extra warp threads, with extra weft threads, and block weave fabric). It is possible to obtain either a lifting plan (Figure 4) or treadling (if it will use the handloom).

So, with just the basic segment and the Woods-Hann code, the ornament itself and all the elements necessary to weave it - the drawing-in draft, and either, the lifting plan or treadling (if it will use the handloom) - can be drawn up using the CAD designed.

Conclusions

This paper presents an adaptation of the Woods-Hann classification and ornament construction system to patterned woven fabrics. It was considered that woven ornaments conform to the general symmetry concepts of the Woods-Hann system, and that it is generally applicable to this kind of ornament. It was determined that only twelve of the seventeen ditranslational symmetry groups can be used for the description of woven fabric ornamentation. There are few limitations in the creation of the woven fabrics: if the centre of rotation or the axis of glide-reflection is laid between columns (rows) or on the column (row), conditionally called threads, the sequence of marked and blank squares of this thread must fulfil certain conditions described in this article; uniform axes (vertical or horizontal) of the ornament with a centred cell must be laid similarly, either on the tread or between threads; any segment of specific form must be used for the creation of ornaments with four-fold rotation: a square for *p411* and *p4gm* symmetry groups, and an isosceles triangle with straight angle for *p4mm* symmetry group. To use the method designed, the multiplication method of matrices is proposed. Every ornament of patterned fabric has its order of horizontal and vertical transformations as well as the basic element. These parts of the presentation of an ornament are in exactly defined relations with the geometrical structures of ornaments in two dimensions as proposed by Woods-Hann. For technological purposes the compact form of ornament presentation can be used. The computer program designed allows the creation of the ornament from an initial segment using any symmetry operation of the Woods-Hann system possible for woven ornaments to see the view of the fabric made by this ornament, and to come from the ornament to the loom setting.

❑

References

1. J. Balèikonis, Audiniu raštai, Valstybiné politinés ir mokslinés literaturos leidykla, Vilnius (1961) (in Lithuanian).
2. A. Tamošaitis, Sodžiaus menas, Žemes ukio rumai, Kaunas (1931) (in Lithuanian).
3. A. Tamošaitis, A. Tamošaitiene, Lithuanian Sashes, Toronto (1988).
4. V. Tumenas, Lietuviu liaudies rinktiniu juostu ornamentas (XVIII a. pab. - XX a. I p.): Tipologija ir semantika, Daktaro disertacija, Kaunas (1997) (in Lithuanian).
5. V. Savoniakaite, Audiniai kaimo kulturoje: lietuviu geometriniai raštai XIX-XX amžiuje, Alma litera, Vilnius (1998) (in Lithuanian).
6. I. Kazlauskiene, D. Neverauskiene, V. Milašius, 'The Symmetry Peculiarities of Designs of National Lithuanian Woven Fabrics - Pick-up Sashes and Overshot Fabrics', Proc. Int. Conf. The Textiles: Research in Design and Technology, Kaunas, Lithuania (September 21 and 22, 2000), pp. 84-92.
7. H.J. Woods, 'The Geometrical Basis of Pattern Design. Part III: Geometrical Symmetry in Plane Patterns', J. Text. Inst., Vol 26, No 1 (1935), pp. T341-T357.
8. M.A. Hann, G.M. Thompson, 'The Geometry of Regular Repeating Patterns', Textile Progress, Vol. 22, No 1 (1992), pp. 1-62.
9. M.A. Hann, 'Symmetry in Two-dimensional Design. A Review of its use in the Analysis and Synthesis of Regular Repeating Patterns', Proc. Int. Conf. The Textiles: Research in Design and Technology, Kaunas, Lithuania (September 21 and 22, 2000), pp. 14-18.

❑ **Received 19.02.2002 Reviewed: 29.04.2002**