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Mathematical Modelling of Mechanical Properties of Textiles Subjected to Tension

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Abstract
Mathematical modelling of textiles subjected to one-directional tensile forces is presented. The models include the straightening of fibres at the beginning, accompanied by the compression of neighbouring fibres, and finally the elongation of the material of fibres when the straightening stops growing. As the result of fibres stretching, the transverse compressive forces hold all the fibres together and are responsible for maintaining the friction forces. Two models are proposed: the rhombus, representing perpendicular fibres, and the helix, representing parallel fibres. The final equations describing both models are found to be the same. Calculations demonstrating the behaviour of the model under dynamic loading were performed. The results are illustrated graphically and discussed.

Key words: *mathematical modelling of textiles, textiles subjected to tenision.*

Introduction

The textiles which are under consideration here are of ordered fibrous structure, which when subjected to tensile forces, causes the fibres constituting the structure to exert transverse pressure on each other, contributing to the development of the friction forces that prevent fibre slippage. These kinds of textiles have wide application in protective equipment, where they are used to reduce hazards. Usually they have the form of ropes [1, 2] or webbings [3]. The webbings are used for making body belts, safety harnesses and lanyards [4-6]. Fall protection equipment is equipped with shock absorbers [7] and fall limiters. To make the deceleration of a falling body smoother, those textiles may be so designed as to make possible elongation larger than is usual. Studies of the behaviour of textiles subjected to a tensile force can be found in papers [8, 9].

Mathematical models

An experimental study [1] of the elongation of ropes under tensile forces showed that the breaking extension of ropes is several times greater than that of the fibres constituting ropes. This is because the fibres are not straight, but have curve form, and the elongation of the fibrous structure at the beginning is primarily a result of fibre straightening and changing fibre configuration. This phenomenon is accompanied by exerting transverse compressive forces on neighbouring fibres, which keep all fibres together and are responsible for maintaining friction forces. When the fibres get locked and hence further elongation due to their straightening becomes insignificant, the extension of the material of the fibres becomes dominant.

In formulating a mathematical model of a fibrous ordered structure subjected to one directional tension, it is symbolically represented by sequences of elements of pitch p. Two types of elements are considered: a rhombus, shown in ***Figure 1***, and a helix, shown in ***Figure 2***. The side of the rhombus is of length $L_p/2$, and that of the helix is equal to L_p. The rhombus represents a structure where fibres under tension are perpendicular to those subjected to compression. The helix represents a structure where both fibres under tension as well as under compression are parallel one to another.

The rhombus is a flat structure and, as we can see from ***Figure 1***, an increase in the pitch p by a value of y results in a decrease in the thickness h by a value of x_h. The helix (***Figure 2***) is a spatial structure and an increase in the pitch p by a value of y causes a decrease in the diameter d by a value of x_d.

For the rhombus (***Figure 1***), the square of the length of its side $L_p/2$ is equal to the sum of the squares of the two sides of the right triangle. Let the compression x_h be a result of the elongation y, without changing the length L_p ***Equation (1)***.

$$\left(\frac{h}{2}\right)^2+\left(\frac{p}{2}\right)^2=\left(\frac{L_p}{2}\right)^2,$$
$$\left(\frac{h}{2}-\frac{x_h}{2}\right)^2+\left(\frac{p}{2}+\frac{y}{2}\right)^2=\left(\frac{L_p}{2}\right)^2, \quad (1)$$
$$(h-x_h)^2+(p+y)^2=L_p^2,$$
$$0\le y\le L_p-p,\ 0\le x_h\le h.$$

The dependence between the tensile force F_y and compressive force F_x can be found from the principle of virtual work ***Equation (2)***.

$$F_y dy - F_x dx_h = 0, F_y \ge 0, F_x \ge 0. \quad (2)$$

The length L_p of the coil of the helix (***Figure 2***) is equal to the diagonal of the rectangle, having sides equal to the circumference πd of the cylinder and its height p. Let the elongation y of the helix be accompanied by the diameter d change of the value of x_d. Introducing the same notation for the helix as previously for the rhombus and using the Pythagorean equation, we can find the dependence between x_d and y ***Equation (3a)***. By substituting d and x_d, this relationship is takes the form ***Equation (3b)***, which is the same as (1).

$$(\pi d)^2+p^2=L_p^2,$$
$$\left(\pi(d-x_d)\right)^2+(p+y)^2=L_p^2. \quad (3a)$$

$$h=\pi d,\ x_h=\pi x_d,$$
$$h^2+p^2=L_p^2, \quad (3b)$$
$$(h-x_h)^2+(p+y)^2=L_p^2.$$

The dependence between the tensile force F_y and compressive pressure f_x can be found from the principle of virtual work ***Equation (4a)***. By substituting f_x and x_d, this relationship takes the form ***Equation (4b)***, which the same as ***Equation (2)***.

$$F_y dy - f_x \frac{dx_d}{2}\pi d = 0. \quad (4a)$$

$$F_x=\tfrac{1}{2}f_x d,\ dx_h=\pi dx_d,$$
$$F_y dy - F_x dx_h = 0. \quad (4b)$$

From ***Equations (1, 3b)*** and ***Equations (2, 4b)*** for both the rhombus and helix, we have the same relationships ***Equation (5)***.

$$x_h = h - \sqrt{L_{p^2} - (p + y)^2},$$
$$0 \leq y \leq L_p - p$$
$$\frac{dx_h}{dy} = \frac{(p + y)}{h - x_h}, 0 \leq x_h < h; \tag{5}$$
$$F_y = F_x \frac{dx_h}{dy},$$

Now we need to have the dependence between the compressive force and the magnitude of compression of fibres $F_x = F_x(x_h)$. It can be taken in from (6), as proposed in paper [10], where it was derived based on the observation that fibres with some curvature undergo flattening under pressure. As a result, the fibres gradually come into mutual contact, which makes their bending length shorter, and thus they become stiffer. In ***Equation (6)*** constants (*k, L*) define the transverse compression stiffness of the element of the fibrous structure due to fibres bending. The values of (*k, L*) for the rhombus are different from those for the helix.

$$F_x = \frac{\mathrm{k}x_h}{\left(1 - \frac{x_h}{L}\right)^3}, \ 0 \leq x_h < L. \tag{6}$$

The extension y_s of fibres, constituting an element of length *p,* under tensile force F_y depends on the material from which they are made and can be expressed by ***Equation (7)***, where c_s is the fibre material's internal damping coefficient and (k_s, k_j) are elasticity parameters defining the longitudinal tensile stiffness of the element of the fibrous structure.

$$F_y = c_s \frac{dy_s}{dt} + k_s y_s + \left(k_j y_s\right)^j \geq 0, \quad y_s \geq 0. \tag{7}$$

The total elongation of the fibrous structure may be sought approximately as a sum of the increase due to the changing fibre configuration *y* and extension of the material of fibres y_s ***Equation (8)***.

$$Y = n_p(y + y_s) \geq 0, \ n_p = \frac{L_B}{p}. \tag{8}$$

In ***Equation (8)*** L_B denotes the length of the fibrous structure and n_p *the* number of elements, shown in ***Figures 1*** or ***2***. Let us denote by *Y* the coordinate of mass *m* attached to the end of the fibrous structure. Let this mass fall down from above the end of the structure. If *Y* becomes negative, then the structure is neither stretched nor compressed, but instead its end is above the position which

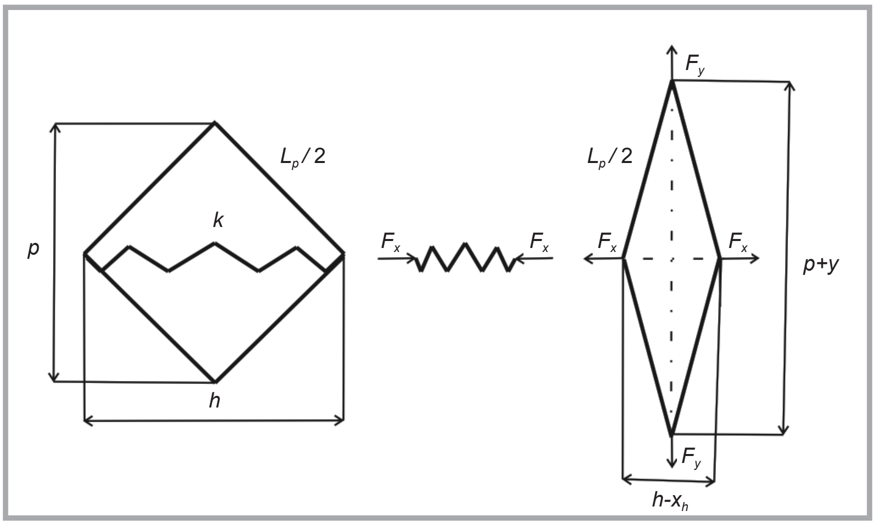

Figure 1. *Rhombus that changes the tensile force F_y into a compressive force F_x, from the left, respectively: non – distorted element, compressed part of the element, scheme of the element for performing calculation in a state of elongation.*

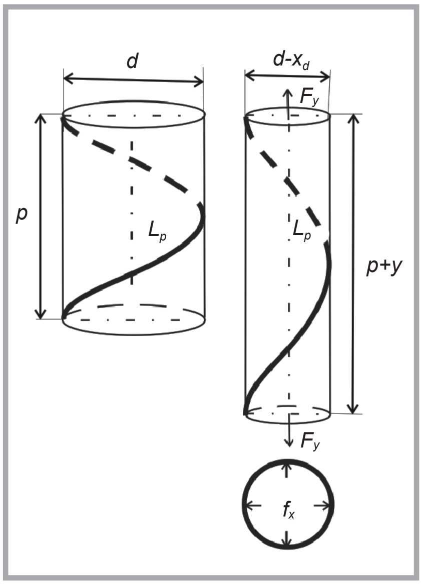

Figure 2. *Coil of a helix that changes the tensile force F_y into compressive pressure f_x; from left, respectively: non- distorted element, scheme of the element for performing calculation in the state of elongation.*

it had at rest. The equation of motion of the mass *m* has the form ***Equation (9)***. Besides the internal damping of the fibre material, there exists damping resulting from the external friction between fibres, which is related to the compressive force F_x and coefficient of the material friction μ. The damping is taken into account by including the friction force μF_x multiplied by the damping coefficient c_x for n_p elements.

$$m\frac{d^2Y}{dt^2} + \mu F_x \mathrm{sgn}\left(\frac{dx_h}{dt}\right) n_p c_x + F_y = \mathrm{mg}. \tag{9}$$

Numerical calculations

In order to investigate the dynamical behaviour of the model proposed, the set of differential equations derived was numerically integrated using the Runge-Kutta method. For this purpose, the following notation was introduced $Y = Y_1$, dY/d$t = Y_2$, $y_s = Y_3$ and the set of ***Equations (5-9)*** was rewritten in the form of algorithm (10). The integration was performed for various parameters,exemplary results of which are shown in ***Figure 3***, obtained for the initial conditions $Y_1(0) = 0$, $Y_2(0) = (2gl_y)^{0.5}$& $Y_3(0) = 0$, the falling mass $m = 60$ kg, gravity acceleration $g = 9.81$ m/s^2, the length of the fibrous structure $L_B = 5$ m, the falling height of the mass attached to the end of the textile structure $l_y = 1$ m, the parameters of the fibrous structure elements $L = 0.006$ m, $d = 0.012$ m, $p = 0.072$ m, $h = \pi d$ & $L_p = (h^2 + p^2)^{0.5}$, the friction coefficient of the fibre material $\mu = 0.2$, the coefficient of the influence of the friction on the damping $c_x = 0.01$, the longitudinal tensile stiffness of the element $k_s = 2700000$ N/m, the nonlinear stiffness coefficient $k_j = 5000$ N$^{1/3}$m^{-1}, nonlinearity exponent $j = 3$, the transverse compression stiffness of the element due to fibre bending $k = 785$ N/m, and the material damping coefficient $c_s = 10000$ Ns/m. Because of the limitations of x_h and y in ***Equations (5)*** and ***(6)***, a sufficiently small time step of the integration must be carefully chosen; herein it was 0.0001s.

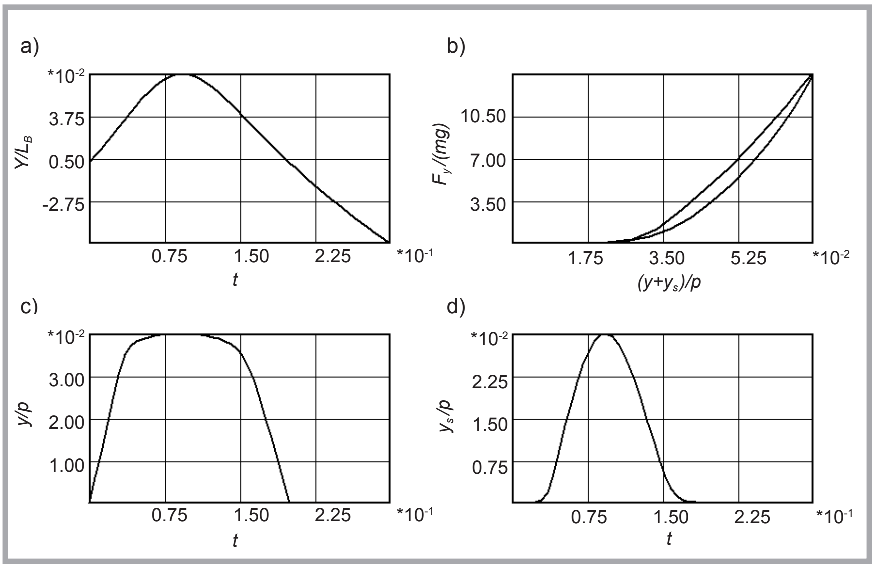

***Figure 3.** a) coordinate Y of the falling mass versus time t [s], b) tensile force F_y versus the total elongation of element $y+y_s$, c) elongation of element y due to the straightening of fibres, and d) the material extension y_s of the element of length p versus time t.*

Discussion of results

A displacement of the mass greater than zero ($Y>0$ ***Figures 3.a***) is equal to the total extension of a fibrous structure of length L_B. When it is negative ($Y<0$), then the mass moves freely and the fibrous structure has its natural length.

From ***Figure 3.b*** we can see that the dependence between the force F_y and the total extension ($y+y_s$) of the fibrous element is nonlinear. Initially nonlinearity is due to the changing fibre configuration, and then with an increase in the elongation, it is due to the nonlinear characteristic of the fibre material. Material damping c_s is responsible for changing the force – elongation curve to an hysteresis loop.

The material extension y_s (***Figure 3.d***) of fibres takes place mainly when the extension y (***Figure 3.c***) due to the straightening of fibres stops increasing. This is because the decrease in the thickness x (***Figures 1*** and ***2***) is close to its limiting value, when the textile structure locks up.

Calculations showed that a decrease in stiffness resulted in an increase in extension Y and a decrease in the tensile force F_y.

$$
\begin{aligned}
& y=\frac{1}{n_p}Y_1-Y_3,\ \text{if}\ \ y<0\ \ \text{then}\ \ y=0,\\
& x_h=h-\sqrt{L_p^2-(p+y)^2},\ \text{if}\ x_h<0\ \text{then}\ x_h=0,\\
& F_x=\frac{\mathrm{k}x_h}{\left(1-\frac{x_h}{L}\right)^3},\\
& \frac{dx_h}{dy}=\frac{(p+y)}{h-x_h},\\
& F_y=F_x\frac{dx_h}{dy},\\
& \frac{dY_1}{dt}=Y_2,\\
& \frac{dY_3}{dt}=\frac{1}{c_s}\left(F_y-k_sY_3-\left(k_jY_3\right)^j\right),\\
& \frac{dy}{dt}=\frac{1}{n_p}Y_2-\frac{\mathrm{d}Y_3}{\mathrm{d}t},\\
& \frac{dx_h}{\mathrm{d}t}=\frac{dx_h}{dy}\frac{dy}{dt},\\
& \text{if}F_y>0\ \ \frac{dY_2}{dt}=\frac{1}{m}\left(\mathrm{mg}-\mu F_x\,\mathrm{sgn}\left(\frac{dx_h}{dt}\right)n_pc_x-F_y\right),\\
& \text{if}F_y\le 0\ \ \frac{dY_2}{dt}=g.
\end{aligned}
\tag{10}
$$

Equation (10).

Conslusions

The equations describing the relation between the tensile force and magnitude of the extension can take the same form for the rhombus model, representing perpendicular fibres, and the helix model, representing parallel fibres.

The mathematical model which is elaborated in this paper can be used for studying properties of the textile part of protective equipment, especially if the studies are followed by suitable experiments.

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